

# Integrable three-state vertex models with weights lying on genus five curves

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## Abstract

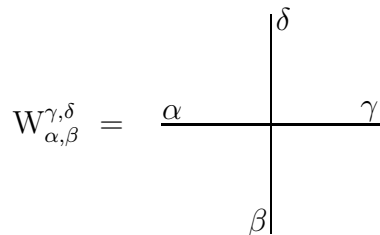
We investigate the Yang-Baxter algebra for  $U(1)$  invariant three-state vertex models whose Boltzmann weights configurations break explicitly the parity-time reversal symmetry. We uncover two families of regular Lax operators with nineteen non-null weights which ultimately sit on algebraic plane curves with genus five. We argue that these curves admit degree two morphisms onto elliptic curves and thus they are bielliptic. The associated R-matrices are non-additive in the spectral parameters and it has been checked that they satisfy the Yang-Baxter equation. The respective integrable quantum spin-1 Hamiltonians are exhibited.

Keywords: Yang-Baxter Equation, Vertex Models, High Genus Curves

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# 1 Introduction

Over the past decades we have witnessed the importance played by vertex models in the development of the theory of integrable systems in two spatial dimensions [1]. Let us recall some of the basic notions about this classical lattice model of statistical mechanics. For simplicity consider a square lattice of size  $N \times N$  with periodic boundary conditions on both horizontal and vertical directions. The statistical configurations are specified by assigning to each set of four edges meeting at a given lattice site the spins variables  $\alpha, \beta, \gamma$  and  $\delta$ . Here we assume that these state variables take values on a finite subset  $V$  of the integers, that is  $V = \{1, \dots, q\}$ . To a given vertex of the lattice we assign a Boltzmann weight  $W_{\alpha, \beta}^{\gamma, \delta}$  representing the energy of the corresponding spins configurations. In figure 1 we illustrated our notation for the weight indices.



The diagram shows a central vertex where four lines intersect, forming a cross. The top vertical line is labeled with the Greek letter  $\delta$ . The bottom vertical line is labeled with the Greek letter  $\beta$ . The left horizontal line is labeled with the Greek letter  $\alpha$ . The right horizontal line is labeled with the Greek letter  $\gamma$ . To the left of this cross, the text  $W_{\alpha, \beta}^{\gamma, \delta} =$  is written, indicating that the cross represents the Boltzmann weight  $W_{\alpha, \beta}^{\gamma, \delta}$ .

Figure 1: The Boltzmann weight of the vertex model on a square lattice.

One feature of the vertex models is that its basic properties can be formulated with the help of a beneath tensor structure formally represented by the product  $V_0 \otimes V_k$  [2]. The horizontal degrees of freedom are encoded in the term  $V_0$  which is often called auxiliary space. The second term  $V_k$  stands for the vertical degrees of freedom associated to each  $k$ -th site of a one dimensional lattice of size  $N$  playing the role of the quantum space of a  $q$ -state spin chain. It turns out that the Boltzmann weights can be organized in terms of a local matrix acting on  $V_0 \otimes V_k$  denominated

Lax operator,

$$L_k(\mathbf{w}) = \sum_{\alpha, \beta, \gamma, \delta=1}^q W_{\alpha, \beta}^{\gamma, \delta} e_{\alpha\gamma}^{(0)} \otimes e_{\beta\delta}^{(k)}, \quad \text{for } k = 1, \dots, N, \quad (1)$$

where  $e_{\alpha\beta}^{(j)}$  denotes  $q \times q$  Weyl matrices acting on the space  $V_j$  with  $j = 0, \dots, N$ . From now on we shall refer to the set of nonzero weights  $W_{\alpha, \beta}^{\gamma, \delta}$  using the symbol “ $\mathbf{w}$ ”.

The respective row-to-row transfer matrix  $T(\mathbf{w})$  can then be written as the trace over the auxiliary space of an ordered product of Lax operators, namely

$$T(\mathbf{w}) = \text{Tr}_{V_0} [L_N(\mathbf{w}) L_{N-1}(\mathbf{w}) \cdots L_2(\mathbf{w}) L_1(\mathbf{w})]. \quad (2)$$

A relevant family of vertex models are those whose Lax operators are invariant by a single  $U(1)$  symmetry. This invariance implies that many of the weights are zero depending on whether or not the respective indices satisfy the so-called ice condition,

$$\begin{aligned} \bullet W_{\alpha, \beta}^{\gamma, \delta} &= 0, \quad \text{for } \alpha + \beta \neq \gamma + \delta \\ \bullet W_{\alpha, \beta}^{\gamma, \delta} &\neq 0, \quad \text{for } \alpha + \beta = \gamma + \delta. \end{aligned} \quad (3)$$

Up to the present the known realizations of vertex models satisfying the above rule have the corresponding weights parameterized in terms of trigonometric functions. The typical examples are the vertex models associated to the solutions of the Yang-Baxter equation based on the quantum  $U[\text{SU}(2)]_{\bar{q}}$  algebra either for generic values of the deformation parameter [3, 4] or when it takes values on the roots of unity [5, 6]. The current results in the literature suggest that to obtain integrable vertex models with weights not uniformized by rational functions one has to consider statistical configurations that violate the  $U(1)$  symmetry. For example, these are the cases of certain generalizations of the eight-vertex model [7] and the celebrated chiral Potts model [8, 9] having both an underlying  $\mathbb{Z}_q$  symmetry.

On the other hand, it has been shown that the transfer matrix of  $U(1)$  invariant vertex models can be diagonalizable by the algebraic Bethe ansatz for arbitrary Lax operators without reference to a given specific parameterization of the weights [10]. In this work it was not needed to make any assumption on the dependence of the spectral parameters entering the corresponding R-matrix to

build up the transfer matrix eigenvectors. In any way this algebraic approach forbids the existence of vertex models having both the  $U(1)$  invariance and Boltzmann weights sitting on algebraic varieties which can not be rationally uniformized. In fact, the generality of a number of weights identities derived in [10] make it hard to believe that they are only realized in terms of trigonometric functions.

Of course, for such class of models, irrationality of weights can only emerge when the number of states  $q$  is larger than two. This is because the model with  $q = 2$  corresponds to the asymmetric six-vertex model whose weights are known to be rationally parameterized, see for example [11]. For  $q > 2$ , however, the structure of the functional relations derived from the Yang-Baxter algebra is very different from that satisfied by weights of an arbitrary six-vertex model. Would they be so stringent to always drive us to exactly solvable vertex models with trigonometric weights?. Here we investigate this question in the most simple case where non-rational weights can not be rule out: the three-state  $U(1)$  vertex model. Another relevant motivation to study these kind of systems comes from the existence of concrete exactly solvable spin-1 quantum chains discovered by Alcaraz and Bariev within the coordinate Bethe ansatz method [12]. The fact that their Hamiltonian for general couplings can not be derived in terms of an additive R-matrix suggests that non rational three-state vertex models should indeed exist.

Our study of Yang-Baxter algebra for  $U(1)$  three-state vertex models will lead us to develop a strategy to deal with a problem involving a large number of functional relations constraining the Boltzmann weights. The basic guidelines of our approach is somehow general and can in principle be used to study more complicated vertex models. It turns out that we are able to uncover two families of integrable  $U(1)$  three-state vertex models with weights lying on non-rational manifolds. In fact, we shall argue that their Boltzmann weights are ultimately constrained by algebraic plane curves of genus five. We remark that the respective quantum spin-1 chains extend in a substantial way the previous integrable Hamiltonian found in reference [12].

We have organized this work as follows. In next section we describe some basic properties of the polynomial relations coming from the Yang-Baxter algebra. We have endeavored to make it self contained when useful mathematical notions of algebraic geometry are used. In section 3 we

describe the main structure of the  $U(1)$  invariant three-state vertex model to be studied in this paper. The solution of the respective functional relations is detailed in Section 4 and we have been able to uncover two families of integrable three-state vertex models. In Section 5 we investigate the geometrical properties underlying the integrability of these vertex models and for one of the families this forces us to analyze the problem of the intersection of two projective surfaces. We found that the properties of the underlying algebraic manifolds are related to that of genus five bielliptic curves. We have summarized the main results for the Lax operators and the respective R-matrix in Section 6. We compute the expressions of the corresponding exactly solvable spin-1 chains and show that they contain as particular case the spin-1 Hamiltonian found previously by Alcaraz and Bariev [12]. Our conclusions are presented in Section 7 and in four Appendices we describe a number of technical details complementing the discussions of the main text.

## 2 Integrability Conditions

In general, a lattice model of statistical mechanics in two-dimensions is considered integrable when the corresponding transfer matrix can be embedded into a family of pairwise commuting operators [1],

$$\left[ T(\mathbf{w}'), T(\mathbf{w}'') \right] = 0, \quad (4)$$

where  $\mathbf{w}'$  and  $\mathbf{w}''$  represent two different sets of weights.

A sufficient condition for commuting transfer matrices was originally introduced by Baxter in his analysis of the eight-vertex model [13]. This condition requires the existence of an invertible R-matrix which together with the Lax operators should satisfy the Yang-Baxter algebra,

$$R(\mathbf{w}', \mathbf{w}'') [L_k(\mathbf{w}') \otimes I_q] [I_q \otimes L_k(\mathbf{w}'')] = [I_q \otimes L_k(\mathbf{w}'')] [L_k(\mathbf{w}') \otimes I_q] R(\mathbf{w}', \mathbf{w}''), \quad (5)$$

where  $I_q$  denotes the  $q \times q$  identity and  $R(\mathbf{w}', \mathbf{w}'')$  is a  $q^2 \times q^2$  matrix acting on the tensor product  $V_0 \otimes V_0$ .

The commutation relation of two distinct transfer matrices (4) is obviously not affected when their weights are multiplied by two independent nonzero scalar factors. This means that the

functional equations coming from Yang-Baxter algebra are expected to be homogeneous separately in each of the sets  $\mathbf{w}'$  and  $\mathbf{w}''$  of weights. To be more precise writing  $F_j(\mathbf{w}', \mathbf{w}'')$  to denote a given polynomial derived from Eq.(5) we then, in general, can state that,

$$F_j(\lambda_1 \mathbf{w}', \lambda_2 \mathbf{w}'') = \lambda_1^{D_1} \lambda_2^{D_2} F_j(\mathbf{w}', \mathbf{w}''), \quad \forall \lambda_1, \lambda_2 \neq 0, \quad (6)$$

where  $D_1$  and  $D_2$  define the bidegree of the bihomogeneous polynomial  $F_j(\mathbf{w}', \mathbf{w}'')$ .

In this paper we will also assume that the R-matrix satisfies the standard unitarity condition,

$$R(\mathbf{w}', \mathbf{w}'') \mathcal{P} R(\mathbf{w}'', \mathbf{w}') \mathcal{P} = \rho(\mathbf{w}', \mathbf{w}'') I_q \otimes I_q \quad (7)$$

where  $\mathcal{P}$  denotes the permutator operator acting on a  $q^2$ -dimensional space and  $\rho(\mathbf{w}', \mathbf{w}'')$  represents an overall normalization.

The above assumption is motivated by the fact that unitarity property (7) assures us from the very beginning that the R-matrix has an inverse. Recall that unitarity has also been relevant in providing us a number of identities that were essential for the algebraic diagonalization of the transfer matrix of the  $U(1)$  invariant vertex models [10]. In addition, we shall show that the unitarity property of the R-matrix imposes an important restriction on the structure of the polynomials  $F_j(\mathbf{w}', \mathbf{w}'')$ . In order to see that we multiply the left and right sides of Eq.(5) by the inverse of the R-matrix and with the help of Eq.(7) we obtain,

$$[L_k(\mathbf{w}') \otimes I_q][I_q \otimes L_k(\mathbf{w}'')] \mathcal{P} R(\mathbf{w}'', \mathbf{w}') \mathcal{P} = \mathcal{P} R(\mathbf{w}'', \mathbf{w}') \mathcal{P} [I_q \otimes L_k(\mathbf{w}'')] [L_k(\mathbf{w}') \otimes I_q]. \quad (8)$$

We now apply the permutator on both sides of Eq.(8) as well as we insert the identity  $\mathcal{P}^2 = I_q \otimes I_q$  in the middle of the brackets to permute the Lax operators. As a result we can derive the following relation,

$$[I_q \otimes L_k(\mathbf{w}')] [L_k(\mathbf{w}'') \otimes I_q] R(\mathbf{w}'', \mathbf{w}') = R(\mathbf{w}'', \mathbf{w}') [L_k(\mathbf{w}'') \otimes I_q] [I_q \otimes L_k(\mathbf{w}')]. \quad (9)$$

Inspecting Eqs.(5,9) we see that their left and right sides are related once we interchange the weights, that is  $\mathbf{w}' \leftrightarrow \mathbf{w}''$ . This means that the polynomial equations coming from the Yang-Baxter algebra are expected to be anti-symmetrical upon the exchange of weights label, that is,

$$F_j(\mathbf{w}', \mathbf{w}'') + F_j(\mathbf{w}'', \mathbf{w}') = 0. \quad (10)$$

We stress that such simple consequence of the unitarity of the R-matrix is going to play an important role to help us disentangle involved high degree functional relations on the Boltzmann weights. It is however fortunate that in many instances of our analysis this fact will come out naturally since we will be able to write the polynomials in the following particular anti-symmetrical form,

$$F_j(\mathbf{w}', \mathbf{w}'') = H_j(\mathbf{w}')G_j(\mathbf{w}'') - H_j(\mathbf{w}'')G_j(\mathbf{w}'), \quad (11)$$

where  $H_j(\mathbf{w})$  and  $G_j(\mathbf{w})$  are irreducible homogeneous polynomials with the same degree  $D$ . We then say that  $F_j(\mathbf{w}', \mathbf{w}'')$  is an irreducible bihomogeneous polynomial with bidegree  $(D, D)$ .

We see that polynomials of the form (11) vanish trivially when we consider the limit  $\mathbf{w}' \rightarrow \mathbf{w}''$  which is a desirable property since certainly the transfer matrix commutes with itself. We next note that such bihomogeneous polynomials always admit an special solution in which the distinct group of weights  $\mathbf{w}'$  and  $\mathbf{w}''$  are decoupled from each other. This solution bears some resemblance with the method of separation of variables used to solve the classical dynamics by the Hamilton-Jacobi theory and partial differential equations of mathematical physics. It can be written as follows,

$$\frac{H_j(\mathbf{w}')}{G_j(\mathbf{w}')} = \frac{H_j(\mathbf{w}'')}{G_j(\mathbf{w}'')} = \Lambda_j, \quad (12)$$

where the parameter  $\Lambda_j$  is considered a free constant.

It turns out that such particular solution to Eq.(11) has a very clear meaning in the realm of algebraic geometry. This discussion permits us to introduce the appropriate mathematical terminology making presentation self contained. We start by recalling that the zero locus of the bihomogeneous polynomial  $F_j(\mathbf{w}', \mathbf{w}'')$  is known to produce a well defined algebraic variety  $X_j$  [14]. In order to define this mathematical object let us denote the set of weights by the elements  $\omega_0, \dots, \omega_m$  representing the coordinates of a projective space  $\mathbb{CP}^m$  over the complex field. The algebraic variety  $X_j$  is a closed subset of the product of such two projective spaces which formally can be represented as,

$$\begin{aligned} X_j &= \{[\omega'_0 : \dots : \omega'_m] \times [\omega''_0 : \dots : \omega''_m] \in \mathbb{CP}^m \times \mathbb{CP}^m \mid H_j(\omega'_0, \dots, \omega'_m)G_j(\omega''_0, \dots, \omega''_m) \\ &\quad - H_j(\omega''_0, \dots, \omega''_m)G_j(\omega'_0, \dots, \omega'_m) = 0\}, \end{aligned} \quad (13)$$

where  $[\omega_0 : \dots : \omega_m]$  denotes a point in the projective space  $\mathbb{CP}^m$  by which we mean the line spanned by the vector  $(\omega_0, \dots, \omega_m) \in \mathbb{C}^{m+1}$  where the origin is omitted.

By the same token, the polynomials originated from the special solution (12) can also be used to define an underlying subvariety  $Y_j \subset X_j$ . This subvariety is in fact described by the product of two identical algebraic sets since the corresponding polynomials do not mix distinct weights labels. We then are able to write  $Y_j = \overline{Y}(\Lambda_j) \times \overline{Y}(\Lambda_j)$  where the component  $\overline{Y}(\Lambda_j)$  is defined by,

$$\overline{Y}(\Lambda_j) = \{[\omega_0 : \dots : \omega_m] \in \mathbb{CP}^m | H_j(\omega_0, \dots, \omega_m) - \Lambda_j G_j(\omega_0, \dots, \omega_m) = 0\} \quad (14)$$

We now can show that the particular solution (12) gives rise to a divisor on the original variety  $X_j$ . To this end we recall that one basic invariant of any variety is its dimension which here can be determined using the standard result that an irreducible hypersurface  $S(w_0, \dots, w_m) \in \mathbb{CP}^m$  has dimension  $\dim S = m - 1$  [14]. From this result it follows that the variety  $X_j$  has dimension  $\dim X_j = 2m - 1$  while the dimension of the subvariety  $Y_j$  is  $\dim Y_j = 2m - 2$  since they are generated by irreducible polynomials. From the fact that  $\dim X_j - \dim Y_j = 1$  and observing that the intersection multiplicity of  $Y_j$  at  $X_j$  is also 1 we then conclude that  $Y_j$  is in fact a prime divisor element on  $X_j$ . This means that by varying the parameter  $\Lambda_j$  we are able to foliate the variety  $X_j$  through submanifolds of codimension 1 whose fibers are determined by the variety  $\overline{Y}(\Lambda_j)$ .

We shall see that the prime divisors associated to polynomials with the structure (11) are precisely the fundamental building blocks of a vertex model with commuting transfer matrix. It is the intersection of a collection of such divisors that ultimately is going to dictate the algebraic variety in which the Boltzmann weights are lying on. This procedure assures us the existence of two independent transfer matrices that are sited on the same algebraic manifold and thus of a single family of Lax operators.

### 3 Three-state Vertex Model

We now turn our attention to the presentation of the specific  $U(1)$  three-state vertex which we intend to investigate in this paper. From the ice-rule (3) it follows that this type of model



can have at most nineteen different Boltzmann weights. This space of parameters can be reduced once we consider typical symmetries of the weights when they are viewed as  $(1+1)$ -dimensional scattering amplitudes [15]. These invariances are defined as follows,

$$\begin{aligned}
&\bullet \text{Parity Reversal : } W_{\alpha,\beta}^{\gamma,\delta} = W_{\beta,\alpha}^{\delta,\gamma}, \\
&\bullet \text{Time Reversal : } W_{\alpha,\beta}^{\gamma,\delta} = W_{\gamma,\delta}^{\alpha,\beta}, \\
&\bullet \text{Charge Conjugation : } W_{\alpha,\beta}^{\gamma,\delta} = W_{q+1-\alpha, q+1-\beta}^{q+1-\gamma, q+1-\delta}.
\end{aligned} \tag{15}$$

According to the recent work [16] nineteen vertex models invariant by the combined action of parity and time reversal symmetries have always rational weights. This means that we have to consider vertex models whose statistical configurations do not preserve the PT transformation. From Eqs.(15) we see that this is achieved when the respective weights fulfill one of the following inequalities,

$$W_{12}^{12} \neq W_{21}^{21}, \quad W_{13}^{13} \neq W_{31}^{31}, \quad W_{23}^{23} \neq W_{32}^{32}, \quad W_{13}^{22} \neq W_{22}^{31}, \quad W_{22}^{13} \neq W_{31}^{22}. \tag{16}$$

One way to assure the breaking of the PT symmetry is by means the diagonal weights  $W_{\alpha,\beta}^{\alpha,\beta}$  since the off-diagonal ones can in principle be modified with the help of gauge transformations. In this case, broken PT invariance is not completely incompatible with the preservation of charge conjugation which in turn permits us to work with a smaller number of distinct weights. Considering that charge symmetry is preserved at least by the diagonal weights our starting ansatz for the Lax operator is,

$$L_k(\mathbf{w}) = \left[ \begin{array}{c|c|c} a e_{11}^{(k)} + b e_{22}^{(k)} + f e_{33}^{(k)} & c e_{21}^{(k)} + d e_{32}^{(k)} & h e_{31}^{(k)} \\ \hline c e_{12}^{(k)} + \bar{d} e_{23}^{(k)} & \bar{b} e_{11}^{(k)} + g e_{22}^{(k)} + \bar{b} e_{33}^{(k)} & d e_{21}^{(k)} + c e_{32}^{(k)} \\ \hline \bar{h} e_{13}^{(k)} & \bar{d} e_{12}^{(k)} + c e_{23}^{(k)} & f e_{11}^{(k)} + b e_{22}^{(k)} + a e_{33}^{(k)} \end{array} \right], \tag{17}$$

where  $a, b, \bar{b}, c, d, \bar{d}, f, g, h$  and  $\bar{h}$  denote ten distinct weights of the set  $\mathbf{w}$ . We see that the PT invariance is only broken by way of the diagonal weights  $b$  and  $\bar{b}$ .

We now consider an arbitrary R-matrix and substitute it together with the above ansatz for the Lax operators in the Yang-Baxter algebra (5). It is not difficult to see that the underlying  $U(1)$

invariance of the Lax operators imposes us severe constraints on the R-matrix. Under the mild assumption that some of the weights of the Lax operators are not trivially related we find that R-matrix elements have also to satisfy the ice-rule (3). This motivates us to choose the R-matrix with the same structure of the Lax operators, namely

$$R(\mathbf{w}', \mathbf{w}'') = \left[ \begin{array}{ccc|ccc|ccc} \mathbf{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{b} & 0 & \mathbf{c} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{f} & 0 & \mathbf{d} & 0 & \mathbf{h} & 0 & 0 \\ \hline 0 & \mathbf{c} & 0 & \bar{\mathbf{b}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\mathbf{d}} & 0 & \mathbf{g} & 0 & \mathbf{d} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\mathbf{b}} & 0 & \mathbf{c} & 0 \\ \hline 0 & 0 & \bar{\mathbf{h}} & 0 & \bar{\mathbf{d}} & 0 & \mathbf{f} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{c} & 0 & \mathbf{b} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{a} \end{array} \right], \quad (18)$$

where bold letters are used to distinguish the R-matrix elements from the Boltzmann weights.

At this point we have the basic ingredients to study the possible solutions of the Yang-Baxter algebra (5). We shall tackle this problem using the following systematic strategy. We start by eliminating the elements of the R-matrix since the main purpose is the determination of weights fixing the Lax operators. To this end we search for suitable functional relations that built a consistent linear system of homogeneous equations for a particular chosen subset of R-matrix entries. The vanishing of respective determinant is going to lead us in most cases to polynomials having the anti-symmetrical structure (11). This makes it possible to define the associated divisors (12) and as a result the freedom of a number of free parameters  $\Lambda_j$ . In the situation of functional relations that can not be written directly in the special form (11) we impose that their polynomials should satisfy the anti-symmetric property (10). This idea is crucial to solve very involved functional equations resulting from many nested steps. It turns out that we always will be able to implement this property at the expense of imposing constraints among the free parameters  $\Lambda_j$ . As a result either the corresponding polynomial vanishes directly or it can be brought into the suitable form (11). After all that, we still have to perform the intersection of some basic divisors which is going

to lead us to the main algebraic manifold for the Boltzmann weights. As a byproduct we are able to determine the matrix elements of the R-matrix in terms of few Lax operators weights. In next sections we show how to carry out all these steps in practice.

## 4 The Functional Relations

The functional equations constraining the entries of the R-matrix and the Boltzmann weights are derived by substituting our proposals (17,18) in the Yang-Baxter algebra. We find that there exists fifty-four independent equations which are best subdivided in terms of their number of distinct terms. In Table 1 we summarize this classification which ranges from relations having only two terms to those with the maximum number of five elements.

Number of Equations	Number of Terms
2	two
15	three
25	four
12	five

Table 1: The number of distinct functional relations versus their respective number of terms.

In general, the technical difficulties in dealing with the solution of the functional equations increase with their number of terms due to the presence of many free variables. However, we shall see that the analysis of the simplest relations with two terms will result in a reasonable decrease of the number of independent equations having three elements. This simplification is important to make possible the solution of such functional equations by the elimination method. In addition to that we will need to analyze only a small number of equations with four terms, as compared to the available set of Table 1, to decide about the integrability of the vertex model we have started with. After this study we are left to verify that all the remaining polynomial equations coming from the Yang-Baxter relation are satisfied. It turns out that this task can be performed algebraically with

the help of computer algebra system.

## 4.1 Two Terms Equations

The functional equations having two terms are given as follows,

$$\begin{aligned}\mathbf{c}(\vec{d}d'' - d'\vec{d}') &= 0, \\ (\bar{\mathbf{d}}d' - \mathbf{d}\vec{d}')c'' &= 0.\end{aligned}\tag{19}$$

We see that the above equations are already in the convenient form (11) since we are disregarding possible solutions with zero weights and R-matrix amplitudes. Clearly, the corresponding divisors fix the following ratios among the variables,

$$\frac{\bar{\mathbf{d}}}{\mathbf{d}} = \frac{\bar{d}}{d} = \Lambda_0.\tag{20}$$

Here we are tacitly assuming that  $\Lambda_0 \neq 0$  since otherwise we would have to set the weight  $\bar{d}$  to zero. Our main interest is to consider genuine nineteen vertex models and thus all the Lax operator weights must be non-null. We now turn to investigate more complicated functional relations having three and four terms.

## 4.2 Three Terms Equations

By considering the ratios (20) the number of independent relations with three elements decrease from the original fifteen to only nine functional equations. Their expressions are given by,

$$\mathbf{a}c'a'' - \bar{\mathbf{b}}c'b'' - \mathbf{c}a'c'' = 0, \quad (21)$$

$$\mathbf{c}c'b'' + \mathbf{b}a'c'' - \mathbf{a}b'c'' = 0, \quad (22)$$

$$\mathbf{a}d'b'' - \mathbf{c}b'd'' - \bar{\mathbf{b}}d'f'' = 0, \quad (23)$$

$$\mathbf{b}b'd'' - \mathbf{a}f'd'' + \mathbf{c}d'f'' = 0, \quad (24)$$

$$\bar{\mathbf{b}}d'a'' - \mathbf{f}d'b'' - \mathbf{d}\bar{b}'c'' = 0, \quad (25)$$

$$\mathbf{d}\bar{d}'\bar{b}'' + \mathbf{f}\bar{b}'c'' - \mathbf{b}f'c'' = 0, \quad (26)$$

$$\mathbf{c}\bar{b}'a'' - \mathbf{c}a'\bar{b}'' - \bar{\mathbf{b}}c'c'' = 0, \quad (27)$$

$$\mathbf{c}f'\bar{b}'' - \mathbf{b}\bar{d}'d'' - \mathbf{c}\bar{b}'f'' = 0, \quad (28)$$

$$\mathbf{d}f'a'' - \mathbf{d}\bar{b}'\bar{b}'' - \mathbf{f}d'c'' = 0. \quad (29)$$

We first observe that Eqs.(21-29) are not invariant when we exchange the R-matrix elements with the corresponding double primed Lax operator weights. The main reason for the absence of this invariance is because we are assuming broken PT symmetry that is  $b \neq \bar{b}$  and  $\mathbf{b} \neq \bar{\mathbf{b}}$ . Therefore, there exists a concrete possibility that the R-matrix and the Lax operators may be sited in two distinct algebraic varieties that are not isomorphic. This is already an indication we have some chance to obtain an integrable vertex model whose weights are not trigonometric.

We now consider the solution of Eqs.(21-29) as a system of homogeneous relations where the unknowns are the R-matrix entries  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\bar{\mathbf{b}}$ ,  $\mathbf{c}$ ,  $\mathbf{d}$  and  $\mathbf{f}$ . We have a number of possibilities of selecting six out of nine equations to construct a consistent linear system for these variables. For a given system to have a non trivial solution the determinant of its coefficients depending on the weights of the Lax operators weights must vanish. We can for example choose Eqs.(21-26) and the corresponding determinant can be written in the following form,

$$\left[ (b'^2 - a'f')c''d'' - (b''^2 - a''f'')c'd' \right] \left[ a'd'c''f'' - b''d''b'c' \right] \left[ \Lambda_0 d'^2 b''\bar{b}'' - c''^2 b'\bar{b}' \right] = 0. \quad (30)$$

From Eq.(30) we see that in principle we have three possible branches to be analyzed depending on the factor we choose to vanish. However, by analyzing other possible systems of six equations we noted that the common factors shared by their determinants are only the first two terms of Eq.(30). As examples of alternative choices of systems we would like to mention those built up from either Eqs.(21-25,29) or Eqs.(21-24,26,29). In any of these cases the form of the third factor always changes and thus it plays the role of an extraneous term. From now on we shall disregard the last factor in Eq.(30) as feasible branch.

We further notice that only the first factor of Eq.(30) has the suitable polynomial form (11). In fact, the second term of Eq.(30) clearly does not vanish as  $\mathbf{w}' \rightarrow \mathbf{w}''$ . This is not an impediment to define an analogous of a divisor but this will lead us to distinct components for each of the weights labels. At this point we can not discard the second factor as a possible branch since we have not yet studied the full properties of the linear system. A more detailed analysis shall reveal us that such apparent asymmetry of the second factor divisor disappears.

#### 4.2.1 Main Branch

This branch is defined by imposing that the first polynomial factor of Eq.(30) is zero. It follows that the corresponding divisor is,

$$\frac{b^2 - af}{cd} = \Lambda_1. \quad (31)$$

We have now the necessary condition to solve Eqs.(21-26) by linear elimination of the variables  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\bar{\mathbf{b}}$ ,  $\mathbf{c}$ ,  $\mathbf{d}$  and  $\mathbf{f}$ . At this stage it is sufficient to present their expressions in a nested form since we have not yet solved the full system of nine equations. By using Eq.(31) for double primed labels

we find that these R-matrix entries can be written as,

$$\frac{\mathbf{a}}{\mathbf{c}} = (b'c'b''d'' - a'd'c''f'')/(\Lambda_1 c'd'c''d''), \quad (32)$$

$$\frac{\bar{\mathbf{b}}}{\mathbf{c}} = (b'c'a''d'' - a'd'b''c'')/(\Lambda_1 c'd'c''d''), \quad (33)$$

$$\frac{\mathbf{b}}{\mathbf{c}} = (\frac{\mathbf{a}}{\mathbf{c}}b'c'' - c'b'')/(a'c''), \quad (34)$$

$$\frac{\mathbf{d}}{\mathbf{c}} = \left[ d'c''(\frac{\bar{\mathbf{b}}}{\mathbf{c}}b'a'' - \frac{\mathbf{b}}{\mathbf{c}}f'b'') \right] / \left[ b'\bar{b}'c''^2 - \Lambda_0 d'^2 b''\bar{b}'' \right], \quad (35)$$

$$\frac{\mathbf{f}}{\mathbf{c}} = \left[ \frac{\mathbf{b}}{\mathbf{c}}\bar{b}'f'c''^2 - \Lambda_0 \frac{\bar{\mathbf{b}}}{\mathbf{c}}d'^2 a''\bar{b}'' \right] / \left[ b'\bar{b}'c''^2 - \Lambda_0 d'^2 b''\bar{b}'' \right], \quad (36)$$

where  $\mathbf{c}$  is an overall normalization.

We see that Eqs(32,33) are singular when the parameter  $\Lambda_1$  is zero and this means that here we have to assume  $\Lambda_1 \neq 0$ . As we shall see the special case  $\Lambda_1 = 0$  will be covered by the branch related to the second polynomial factor of the determinant (30). Let us now begin the analysis of the remaining three relations considering first Eq.(27). After substituting the expression for the ratio  $\bar{\mathbf{b}}/\mathbf{c}$  (33) in Eq.(27) we easily find that it becomes proportional to the polynomial,

$$(b'c' - \Lambda_1 \bar{b}'d')a''d'' - (b''c'' - \Lambda_1 \bar{b}''d'')a'd' = 0, \quad (37)$$

which again has the form (11) and the associated divisor is,

$$\frac{bc - \Lambda_1 \bar{b}d}{ad} = \Lambda_2. \quad (38)$$

We next consider the solution of Eq.(28). We first observe that we have already been able to reduce the number of independent variables by two weights. In fact, considering Eqs.(31,38) it is not difficult to resolve the weights  $f$  and  $d$  in terms of the remaining variables  $a, b, \bar{b}$  and  $c$ . By using this information together with the expression for the ratio  $\mathbf{b}/\mathbf{c}$  (34) and after few simplifications we find that Eq.(28) can be expressed as follows,

$$\begin{aligned} & \left[ \Lambda_2 a'^2 \bar{b}' + \Lambda_1 a' \bar{b}'^2 - \frac{\Lambda_0}{\Lambda_1} a' b'^2 \right] \left[ \Lambda_2 a'' b''^2 + \Lambda_1 b''^2 \bar{b}'' - \Lambda_1 b'' c''^2 \right] \\ & - \left[ \Lambda_2 a''^2 \bar{b}'' + \Lambda_1 a'' \bar{b}''^2 - \frac{\Lambda_0}{\Lambda_1} a'' b''^2 \right] \left[ \Lambda_2 a' b'^2 + \Lambda_1 b'^2 \bar{b}' - \Lambda_1 b' c'^2 \right]. \end{aligned} \quad (39)$$

Notice that the polynomial (39) is already written in the form (11) . Its dependence on the parameter  $\Lambda_0$  can be re-scaled by means of the following transformation,

$$\Lambda_1 = \tilde{\Lambda}_1 \sqrt{\Lambda_0} \quad \text{and} \quad \Lambda_2 = \tilde{\Lambda}_2 \sqrt{\Lambda_0}, \quad (40)$$

and now Eq.(39) becomes only dependent on  $\tilde{\Lambda}_1$  and  $\tilde{\Lambda}_2$ . This re-scaling of parameters is ultimately due to the freedom of implementing a gauge transformation on the weights  $d$  and  $\bar{d}$ . Taking into account Eq.(40) we find that the respective divisor associated to the polynomial (39) is given by,

$$\frac{a \left[ b^2 - \tilde{\Lambda}_1 \tilde{\Lambda}_2 a \bar{b} - \tilde{\Lambda}_1^2 \bar{b}^2 \right]}{b \left[ \tilde{\Lambda}_2 a b + \tilde{\Lambda}_1 b \bar{b} - \tilde{\Lambda}_1 c^2 \right]} = \Lambda_3. \quad (41)$$

We have now reached a point in which only Eq.(29) remains to be solved. As before we would like to use the last divisor (41) to eliminate one further weight and thus reducing the number of degrees of freedom. We observe however that the divisor (41) does not provide us the means to eliminate any weight in a linear way. This difficult can be circumvented once we analyze the polynomial associated to Eq.(29) and notice that it depends on the weight  $c$  only through even powers. This means that we can use the fact that divisor (41) has a quadratic dependence on the weight  $c$  to systematically eliminate this weight from Eq.(29). In Appendix A we explain how this operation can be implemented within the Mathematica algebraic computer system. As a result of this procedure we find an involved bihomogeneous polynomial with bidegree (4,4) which unfortunately can not be brought into the convenient form (11). At this stage of the analysis the requirement that the polynomials must satisfy the anti-symmetric property (10) becomes decisive to make further progress. By imposing this property we find that it can indeed be fulfilled provided that the so far free parameters are constrained by the following simple relation,

$$\Lambda_3 \tilde{\Lambda}_2 - \tilde{\Lambda}_1^2 - 1 = 0. \quad (42)$$

After using the condition (42) we find that a large number of terms of the resulting polynomial coming from Eq.(29) are magically canceled out. Thanks to this simplification we are able to write the resulting polynomial in the appropriate form (11). In what follows we shall present the



respective divisor since from it one can easily recover the associated polynomial. The expression for the divisor is given by,

$$\frac{\tilde{\Lambda}_2 a^2 (\tilde{\Lambda}_2^2 \bar{b}^2 - \Lambda_3^2 b^2) + \tilde{\Lambda}_1 \tilde{\Lambda}_2 \bar{b}^3 (2\tilde{\Lambda}_2 a + \tilde{\Lambda}_1 \bar{b}) + b^2 (\Lambda_3^3 \bar{b}^2 - \tilde{\Lambda}_2 b^2)}{\bar{b} b^2 (\tilde{\Lambda}_1 \bar{b} + \tilde{\Lambda}_2 a)} = \Lambda_4. \quad (43)$$

This completes the solution of the nine functional relations for this branch. Up to this point the effective integrable manifold of the weights should be given by the intersection of the last two divisors (41,43). In addition to that we have four free parameters at our disposal since so far we just have the constrain (42). These divisors give rise to two projective surfaces  $\in \mathbb{CP}^3$  and is generally expected that their intersection will lead us to an algebraic spatial curve. Since some of the free parameters are going to be fixed later on we shall postpone the analysis of the intersection until the very end.

#### 4.2.2 Special Branch

This branch is defined by setting the second term of Eq.(30) to zero and it is related to the particular value  $\Lambda_1 = 0$  excluded in the main branch. This can be seen by first noticing that the R-matrix ratio  $\mathbf{a}/\mathbf{c}$  (32) is proportional to the second factor of the determinant (30). For this ratio to be no null we should have another zero on the denominator such that the value of the ratio becomes indeterminate. Inspecting Eq.(32) we note that the only option is to set  $\Lambda_1 = 0$  and as result it follows that the weight  $f$  must be fixed by the expression,

$$f = \frac{b^2}{a}. \quad (44)$$

We now observe that the numerator of Eq.(33) must vanish otherwise we would have a divergence on the ratio  $\bar{\mathbf{b}}/\mathbf{c}$ . From this condition it follows a polynomial relation of the form (11), namely

$$b' c' a'' d'' - b'' c'' a' d' = 0, \quad (45)$$

whose corresponding divisor is,

$$\frac{bc}{ad} = \Lambda_2. \quad (46)$$

The next step is to evaluate the indeterminacy of the above mentioned R-matrix ratios. This is fortunately done with the help of Eqs.(21,27) and the final expressions for such ratios are,

$$\mathbf{a}/\mathbf{c} = [\bar{b}' a'' b'' - a' b'' \bar{b}'' + a' c'' a'']/(c' c'' a''), \quad (47)$$

$$\bar{\mathbf{b}}/\mathbf{c} = (\bar{b}' a'' - a' \bar{b}'')/(c' c''), \quad (48)$$

while the other R-matrix entries can again be computed from Eqs.(34-36).

At this stage the only relations that remain to be solved are Eqs.(28,29). The technical details entering their solution are fairly parallel to those already explained in the previous subsection for the main branch. In what follows we shall therefore present only the final results for the corresponding divisors. We find that the divisor associated to Eq.(28) has the following form,

$$\frac{\tilde{\Lambda}_2 a^2 \bar{b}}{bc^2 - b^2 \bar{b}} = \Lambda_3, \quad (49)$$

where  $\Lambda_2$  has been re-scaled as in Eq.(40) and together with the parameter  $\Lambda_3$  they satisfy the constraint,

$$\Lambda_3 \tilde{\Lambda}_2 - 1 = 0. \quad (50)$$

Finally, the divisor associated to the anti-symmetrical polynomial derived from Eq.(29) is given by,

$$\frac{\tilde{\Lambda}_2 a^2 (\tilde{\Lambda}_2^2 \bar{b}^2 - \Lambda_3^2 b^2) + b^2 (\Lambda_3^3 \bar{b}^2 - \tilde{\Lambda}_2 b^2)}{\tilde{\Lambda}_2 \bar{b} b^2} = \Lambda_4. \quad (51)$$

We note that the divisors (44,46,51) can be directly obtained from those derived for the main branch by substituting  $\tilde{\Lambda}_1 = 0$  in Eqs.(31,38,43), respectively. The same observation for the remaining divisor (49) is more subtle since the zero order in  $\tilde{\Lambda}_1$  of the corresponding main branch divisor (41) is trivial. In spite of that we can recover the divisor (49) by rewriting Eq.(41) in powers of the parameter  $\tilde{\Lambda}_1$ . By setting the coefficient proportional to  $\tilde{\Lambda}_1$  to zero we then easily obtain the divisor (41). In practice, since this limit is somewhat delicate we shall consider these two branches separately.

### 4.3 Four Terms Equations

The Boltzmann weights  $h$ ,  $\bar{h}$  and  $g$  begin to emerge only in the functional equations having four distinct terms. Even after using the divisor (20) we find that the total number of relations with four terms is still considerable. Altogether we have twenty-two functional equations which is a high number to approach the problem by the standard elimination method. As we shall see however we need to solve only ten functional relations to decide on the integrability of the vertex model. The reason for this simplification is that few of the equations with four terms have as unknowns the R-matrix entries  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\bar{\mathbf{b}}$  which have already been fixed. For sake of consistent of the elimination procedure we are then forced to make linear combinations between a small subset of four terms functional equations and two particular three terms relations. Remarkably enough this simple reasoning is able to determine the structure of the remaining weights  $h$ ,  $\bar{h}$  and  $g$ . We shall first detail how this procedure works for the weight  $g$ .

#### 4.3.1 The weight $g$

We have three functional equations depending uniquely on the weight  $g$  and on some variables that have been previously determined. Their expressions are given by,

$$\mathbf{c}g'b'' + \mathbf{b}c'c'' - \bar{\mathbf{b}}\bar{d}'d'' - \mathbf{c}b'g'' = 0, \quad (52)$$

$$\mathbf{g}d'\bar{b}'' + \mathbf{d}b'c'' - \mathbf{c}\bar{b}'d'' - \mathbf{b}d'g'' = 0, \quad (53)$$

$$\mathbf{g}\bar{b}'c'' + \mathbf{d}\bar{d}'b'' - \mathbf{c}c'\bar{b}'' - \bar{\mathbf{b}}g'c'' = 0. \quad (54)$$

We see that Eqs.(52-54) have five unknowns and therefore we need more two equations to build up a consistent homogeneous linear system. These extra relations should be searched among those solved in subsection (4.2.1) since the above equations depend on the weights  $\mathbf{b}$ ,  $\bar{\mathbf{b}}$  and  $\mathbf{c}$ . Direct inspection of Eqs.(21-29) reveals us that this choice is remarkably unique once we want to keep the minimal number of five unknowns. These suitable relations turn out to be Eqs.(27,28). Now, by setting the determinant of equations (27,28,52-54) equal to zero we find that it can be factorized

as,

$$\begin{aligned} & \left[ \Lambda_0(b' c' d' c'' d'' g'' - b'' c'' d'' c' d' g') + \Lambda_0^2(\bar{b}' d'^2 a'' d''^2 - \bar{b}'' d''^2 a' d'^2) \right. \\ & \left. + \bar{b}' c'^2 c''^2 f'' - \bar{b}'' c''^2 c'^2 f' \right] \left[ b' \bar{b}' c''^2 - \Lambda_0 d'^2 b'' \bar{b}'' \right] = 0. \end{aligned} \quad (55)$$

The determinant (55) must vanish through the first factor since the second one is exactly the extraneous term that has been discarded before. We see that the first factor of Eq.(55) is anti-symmetrical on the exchange of weights labels and taking into account the divisors (31,38,41) we are indeed able to rewrite it in the appropriate form (11). The resulting polynomial depends only on the weights  $a, b, \bar{b}, g$  and the corresponding divisor is given by,

$$\frac{\tilde{\Lambda}_2 \Lambda_3 b (\tilde{\Lambda}_2 a + \tilde{\Lambda}_1 \bar{b}) g - \bar{b} [\Lambda_3 b^2 + \tilde{\Lambda}_2 (\tilde{\Lambda}_2 a + \tilde{\Lambda}_1 \bar{b})^2]}{b^2 (\tilde{\Lambda}_2 a + \tilde{\Lambda}_1 \bar{b})} = \Lambda_5 \quad (56)$$

Let us consider the solution of the remaining equations (53,54). We can easily solve one of these relations fixing the value of the R-matrix element  $\mathbf{g}$ . For instance from Eq.(53) we find,

$$\frac{\mathbf{g}}{\mathbf{c}} = \left[ \frac{\mathbf{b}}{\mathbf{c}} d' g'' + \bar{b}' d'' - \frac{\mathbf{d}}{\mathbf{c}} b' c'' \right] / (d' \bar{b}'') \quad (57)$$

where the ratios  $\mathbf{b}/\mathbf{c}$  and  $\mathbf{d}/\mathbf{c}$  have already been determined by Eqs.(34,35).

The last equation (54) can now be rewritten only in terms of the weights  $a, b$  and  $\bar{b}$  once we use the help of the divisors (31,38,41,56). After some simplifications we find that Eq.(54) becomes proportional to the following expression,

$$\begin{aligned} & \left[ \Lambda_5 - \tilde{\Lambda}_1 (\Lambda_3 + \tilde{\Lambda}_2) \right] \left[ b''^2 (\tilde{\Lambda}_2 \Lambda_3^2 a'^2 + \tilde{\Lambda}_2 b'^2 + \tilde{\Lambda}_2 \Lambda_5 a' \bar{b}' + (\tilde{\Lambda}_1 \Lambda_5 - \Lambda_3^3) \bar{b}'^2) + \right. \\ & \left. \tilde{\Lambda}_2 (\Lambda_3 - \tilde{\Lambda}_2) a'' \bar{b}'' (\tilde{\Lambda}_2 a' \bar{b}' + \tilde{\Lambda}_1 \bar{b}'^2) - \tilde{\Lambda}_2 \bar{b}''^2 (\tilde{\Lambda}_2 \Lambda_3 a'^2 + \tilde{\Lambda}_1 (\Lambda_3 + \tilde{\Lambda}_2) a' \bar{b}' + \tilde{\Lambda}_1^2 \bar{b}'^2) \right]. \end{aligned} \quad (58)$$

The polynomial (58) is very far from satisfying the anti-symmetrical property (10) but nevertheless it is proportional to a combination of free parameters. We then are able to solve Eq.(54) by choosing that its first factor vanishes and as result the parameter  $\Lambda_5$  becomes fixed by,

$$\Lambda_5 = \tilde{\Lambda}_1 (\Lambda_3 + \tilde{\Lambda}_2) \quad (59)$$

We finally remark that though the above discussion has been detailed for main branch there is no difficulty to repeat the same analysis in the case of the special branch. It turns out that for the special branch we simply have to set  $\tilde{\Lambda}_1 = 0$  in the final results (56,59).

### 4.3.2 Weights $h$ and $\bar{h}$

Out of thirteen functional equations involving the weights  $h$  and  $\bar{h}$  we shall need only seven of them to determine both these weights and the R-matrix elements  $\mathbf{h}$  and  $\bar{\mathbf{h}}$ . The expressions of such basic relations are,

$$\mathbf{c}h'\bar{b}'' + \bar{\mathbf{b}}c'c'' - \mathbf{b}d'd'' - \mathbf{c}\bar{b}'h'' = 0, \quad (60)$$

$$\mathbf{c}\bar{h}'\bar{b}'' + \bar{\mathbf{b}}c'c'' - \mathbf{b}\bar{d}'\bar{d}'' - \mathbf{c}\bar{b}'\bar{h}'' = 0, \quad (61)$$

$$\mathbf{h}b'c'' - \mathbf{b}h'c'' - \mathbf{c}c'b'' + \mathbf{d}d'\bar{b}'' = 0, \quad (62)$$

$$\bar{\mathbf{h}}b'c'' - \mathbf{b}\bar{h}'c'' - \mathbf{c}c'b'' + \bar{\mathbf{d}}\bar{d}'\bar{b}'' = 0, \quad (63)$$

$$\mathbf{h}\bar{d}'b'' + \mathbf{d}\bar{b}'c'' - \mathbf{c}b'd'' - \bar{\mathbf{b}}d'\bar{h}'' = 0, \quad (64)$$

$$\bar{\mathbf{h}}d'b'' + \bar{\mathbf{d}}\bar{b}'c'' - \mathbf{c}b'd'' - \bar{\mathbf{b}}\bar{d}'h'' = 0, \quad (65)$$

$$\mathbf{a}h'a'' - \mathbf{d}c'd'' - \mathbf{f}h'f'' - \mathbf{h}a'h'' = 0. \quad (66)$$

As before we observe that Eqs.(60,61) have as unknowns the R-matrix elements  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\bar{\mathbf{b}}$  and these equations can again be solved by making linear combinations with Eqs.(27,28). By requiring that the determinant of Eqs.(27,28,60) vanishes we obtain,

$$c'd'c''d'' \left[ \bar{b}'(h'' - a'' - f''/\Lambda_0) - \bar{b}''(h' - a' - f'/\Lambda_0) \right] = 0, \quad (67)$$

while the vanishing of the one made by system of equations (27,28,61) is,

$$c'd'c''d'' \left[ \bar{b}'(\bar{h}'' - a'' - \Lambda_0 f'') - \bar{b}''(\bar{h}' - a' - \Lambda_0 f') \right] = 0. \quad (68)$$

The second factors of the above relations give rise to anti-symmetrical polynomials of the form (11) and their corresponding divisors determine the expressions for the weights  $h$  and  $\bar{h}$ ,

$$\frac{h - a - f/\Lambda_0}{\bar{b}} = \Lambda_6 \quad \text{and} \quad \frac{\bar{h} - a - \Lambda_0 f}{\bar{b}} = \bar{\Lambda}_6 \quad (69)$$

We next note that from Eqs.(62,63) we can retrieve the R-matrix entries  $\mathbf{h}$  and  $\bar{\mathbf{h}}$ . We further observe that the last two terms of these equations are also present in the previously solved three

terms relations (22,26). This makes it possible to eliminate the terms  $\mathbf{c}\mathbf{c}'\mathbf{b}''$  and  $\mathbf{d}\mathbf{d}'\mathbf{b}''$  of Eqs.(62,63) and after using the divisors (69) for the single primed labels we find that the ratios  $\mathbf{h}/\mathbf{c}$   $\bar{\mathbf{h}}/\mathbf{c}$  are given by,

$$\frac{\mathbf{h}}{\mathbf{c}} = \frac{\mathbf{a}}{\mathbf{c}} + \left(\frac{\mathbf{f}}{\mathbf{c}}\right)/\Lambda_0 + \Lambda_6 \frac{\bar{b}' \mathbf{b}}{b' \mathbf{c}}, \quad \text{and} \quad \frac{\bar{\mathbf{h}}}{\mathbf{c}} = \frac{\mathbf{a}}{\mathbf{c}} + \Lambda_0 \left(\frac{\mathbf{f}}{\mathbf{c}}\right) + \bar{\Lambda}_6 \frac{\bar{b}' \mathbf{b}}{b' \mathbf{c}} \quad (70)$$

We now have the basic ingredients to tackle the solution of Eqs.(64,65). Considering the above results altogether as well as the previous expressions for the ratios  $\mathbf{d}/\mathbf{c}$  and  $\bar{\mathbf{b}}/\mathbf{c}$  we are able to write these equations solely in terms of the weights  $a$ ,  $b$  and  $\bar{b}$ . We find that the vanishing of Eqs.(64,65) are equivalent to the the following identities,

$$\Lambda_6 \Lambda_0 \bar{b}' \left[ \tilde{\Lambda}_2(b'^2 a'' \bar{b}'' - b''^2 a' \bar{b}') + \tilde{\Lambda}_1(b'^2 \bar{b}''^2 - b''^2 \bar{b}'^2) \right] + \bar{\Lambda}_6 \Lambda_3 b'^2 \bar{b}'' (a' \bar{b}'' - a'' \bar{b}') = 0, \quad (71)$$

$$\bar{\Lambda}_6 \bar{b}' \left[ \tilde{\Lambda}_2(b'^2 a'' \bar{b}'' - b''^2 a' \bar{b}') + \tilde{\Lambda}_1(b'^2 \bar{b}''^2 - b''^2 \bar{b}'^2) \right] + \Lambda_6 \Lambda_3 \Lambda_0 b'^2 \bar{b}'' (a' \bar{b}'' - a'' \bar{b}') = 0. \quad (72)$$

We see that Eq.(71,72) contain common anti-symmetrical polynomials which in principle could be set to zero by means of the corresponding divisors. This possibility imposes at least an extra constrain on the weights  $a$  and  $\bar{b}$  in addition to the other two divisors (41,43) already derived in subsection (4.2.1). The intersection of such three divisors will generically lead us to zero dimensional manifold consisted of finite number of points for the ratios  $a/c$ ,  $b/c$  and  $\bar{b}/c$  and thus to Lax operators without free spectral parameters. From now on we shall discard this kind of possible “braid” solutions of the Yang-Baxter algebra. However, it is fortunate that Eqs.(71,72) can vanish without the definition of any additional divisor by choosing the last two free parameters to be zero, that is,

$$\Lambda_6 = \bar{\Lambda}_6 = 0 \quad (73)$$

Let us finally consider the solution of Eq.(66). Considering all the results we have obtained so far we find that Eq.(66) leads us once again to deal with a very complicated polynomial of bidegree (4,4) on the weights  $a$ ,  $b$  and  $\bar{b}$ . To make progress we then impose the anti-symmetrical property (10) in the expectation of further simplifications. Fortunately, we find that such property can be satisfied provided we introduce a new constrain among the parameters, namely

$$\Lambda_4 \tilde{\Lambda}_1 \Lambda_0 + \Lambda_3^2 \left[ \tilde{\Lambda}_2(1 + \Lambda_0^2) - \Lambda_3 \Lambda_0 \right] - \tilde{\Lambda}_2 \Lambda_0 (1 - \tilde{\Lambda}_1^2) = 0. \quad (74)$$

By substituting the constrain (74) back into Eq.(66) we find that its non trivial terms become proportional to the following polynomial,

$$(\Lambda_0^2 - \Lambda_0 + 1)(a' \bar{b}'' - a'' \bar{b}') \left[ \tilde{\Lambda}_1 a' (b'^2 + \bar{b}'^2) + \bar{b}' (\tilde{\Lambda}_2 a'^2 + \Lambda_3 b'^2) \right] \left[ \tilde{\Lambda}_1 a'' (b''^2 + \bar{b}''^2) + \bar{b}'' (\tilde{\Lambda}_2 a''^2 + \Lambda_3 b''^2) \right]. \quad (75)$$

Taking into account the above discussion we conclude that Eq.(66) should vanishes by imposing that the first factor of (75) is zero. As a consequence the parameter  $\Lambda_0^2$  is constrained to take values on the non trivial third order roots of unity, namely

$$\Lambda_0 = \exp(\pm i \frac{\pi}{3}). \quad (76)$$

Now by considering the above constrain for  $\Lambda_0$  back to Eq.(74) we see that this parameter factorizes leading us to an effective condition independent of  $\Lambda_0$ ,

$$\Lambda_4 \tilde{\Lambda}_1 + \Lambda_3^2 (\tilde{\Lambda}_2 - \Lambda_3) - \tilde{\Lambda}_2 (1 - \tilde{\Lambda}_1^2) = 0. \quad (77)$$

Although the above discussion has been concentrated in the case of the main branch the same reasoning applies straightforwardly for the special branch. Once again we just have to impose  $\tilde{\Lambda}_1 = 0$  in the final results and we see that in practice this becomes relevant only for Eq.(77). By substituting  $\tilde{\Lambda}_1 = 0$  in this relation and with the help of Eq.(50) we conclude that for the special branch the parameter  $\tilde{\Lambda}_2$  is required to satisfy the relation  $\tilde{\Lambda}_2^4 - \tilde{\Lambda}_2^2 + 1 = 0$ . This means that the for special branch all the parameters have already been fixed with the exception of  $\Lambda_4$ . By way of contrast for the main branch we have two free parameters and to avoid division by a potential zero factor we choose them to be  $\tilde{\Lambda}_2$  and  $\Lambda_3$ . For sake of clearness we have summarized the values of fixed parameters for both branches in Table 2.

Parameters	$\Lambda_0$	$\tilde{\Lambda}_1$	$\tilde{\Lambda}_2$	$\Lambda_3$	$\Lambda_4$	$\Lambda_5$	$\Lambda_6, \bar{\Lambda}_6$
Main Branch	$\exp(\pm i \frac{\pi}{3})$	$\pm \sqrt{\tilde{\Lambda}_2 \Lambda_3 - 1}$	Free	Free	$\frac{\Lambda_3^2 (\Lambda_3 - \tilde{\Lambda}_2) + \tilde{\Lambda}_2 (1 - \tilde{\Lambda}_1^2)}{\tilde{\Lambda}_1}$	$\tilde{\Lambda}_1 (\Lambda_3 + \tilde{\Lambda}_2)$	0
Special Branch	$\exp(\pm i \frac{\pi}{3})$	0	$\pm \Lambda_0^{\pm \frac{1}{2}}$	$\tilde{\Lambda}_2^{-1}$	Free	0	0

Table 2: The values of the parameters associated to the divisors for both branches.

We now reach a point in which there are twenty-four functional relations involving four and five terms that still have to be verified. To avoid overcrowding this section with a number of additional formulae we have presented their explicit expressions in Appendix A. Taking into account the results obtained so far the bihomogeneous polynomials associated to Eqs.(A.1-A.24) can clearly be expressed in terms of the weights  $a$ ,  $b$ ,  $\bar{b}$  and  $c$ . In the case of the main branch we can use the help of the divisors (41,43) to simplify these polynomials in a systematic way until we reach the point where the weight  $c$  is completely eliminated and the powers of the weights  $\bar{b}$  are at most three. Remarkably enough, after such algebraic manipulations these twenty-four polynomials are either zero or become proportional to the factor  $\Lambda_0^2 - \Lambda_0 + 1$  and consequently vanish thanks to the constrain (76). Similar reasoning can be implemented for the special branch by considering now the respective divisors (49,51). The technical details concerning these simplifications are described in Appendix A including the explicit computer algebra system code used. In this way we are able to verify **algebraically** that the whole Yang-Baxter algebra (5) is indeed satisfied for both branches. We emphasize that this verification can be done in rather modest computer as far as memory is concerned and without the need to take any a priori numeric values either for the available free parameters or for the Boltzmann weights.

## 5 The Manifolds Geometry

The purpose of this section is to present the structure of the main algebraic manifolds governing the integrability of the previously uncovered vertex models as well as to discuss their geometric properties. We shall argue that the weights of the Lax operators can be written solely in terms of three variables  $a$ ,  $b$  and  $c$  which are constrained by projective plane algebraic curves of genus five for generic values of the free parameters. Since we will be dealing mostly with singular manifolds we start by recalling the definition of singular locus of an irreducible hypersurface  $S(\omega_0, \dots, \omega_m) \in \mathbb{CP}^m$ . The set of singular points of this hypersurface form a closed subvariety  $\text{Sing}(S)$  determined by the zeroes of all the partial derivatives of  $S(\omega_0, \dots, \omega_m)$ ,

$$\text{Sing}(S) = \{[\omega_0 : \dots : \omega_m] \in \mathbb{CP}^m \mid \frac{\partial S}{\partial \omega_j} = 0, \text{ for } j = 0, \dots, m.\} \quad (78)$$



The analysis of the singularities is going to be helpful to elucidate the intersection problem of surfaces in the case of the main branch. This study is also essential for the understanding of the geometric properties of the algebraic curves we shall confront here.

## 5.1 The Main Branch

From the majority of the divisors we are able to extract the weights in terms of the variables  $a$ ,  $b$  and  $c$  by means of linear elimination. The only exception for this branch is the weight  $\bar{b}$  since we have to deal with the intersection of the divisors (41,43) which are both non-linear in this variable. Before considering this problem we find convenient to perform the following re-scaling on the parameters  $\tilde{\Lambda}_2$  and  $\Lambda_3$ ,

$$\tilde{\Lambda}_2 = \tilde{\Lambda}_1 \varepsilon_1 \quad \text{and} \quad \Lambda_3 = \tilde{\Lambda}_1 \varepsilon_2. \quad (79)$$

where  $\varepsilon_1$  and  $\varepsilon_2$  are now the two free parameters associated to the main branch.

Considering the above re-scaling and the parameters values of Table (2) we can rewrite the divisors (41,43) as two projective surfaces in  $\mathbb{CP}^3$ . The polynomial expression associated to divisor (41) becomes,

$$S_1(a, b, \bar{b}, c) = a(b^2 + \bar{b}^2) + (\varepsilon_1 a^2 + \varepsilon_2 b^2) \bar{b} - \varepsilon_2 b c^2, \quad (80)$$

while the divisor (43) can be rewritten as,

$$\begin{aligned} S_2(a, b, \bar{b}) &= a^2(\varepsilon_1^2 \bar{b}^2 - \varepsilon_2^2 b^2) + (2\varepsilon_1 a + \bar{b}) \bar{b}^3 - (\varepsilon_1 \varepsilon_2 - \varepsilon_2^2 - 2) b^2 \bar{b}^2 - (\varepsilon_1 \varepsilon_2 - 1) b^4 \\ &- (\varepsilon_1^2 \varepsilon_2 - \varepsilon_1 \varepsilon_2^2 - 2\varepsilon_1 + \varepsilon_2^3) a b^2 \bar{b}. \end{aligned} \quad (81)$$

The surface  $S_1$  is free of singularities for generic values of the parameters  $\varepsilon_1$  and  $\varepsilon_2$  because there is no solution to Eq.(78) other than the origin. This is a rational manifold since any nonsingular cubic surface is known to be birational to  $\mathbb{CP}^2$  [14]. The surface  $S_2$  does not depend on the weight  $c$  and thus has to be seen as cone over a plane algebraic curve. By solving Eq.(78) for  $S_2$  we find that it contains two non coplanar singular lines given by,

$$\mathbf{l}_1 = [a : 0 : 0 : 1] \quad \text{and} \quad \mathbf{l}_2 = [a : 0 : -\varepsilon_1 a : 1], \quad (82)$$

and for this reason it is expected that  $S_2$  should be a cone over an elliptic curve [17]. In appendix B we present the technical details showing that  $S_2$  indeed defines a curve of genus one when viewed in the ring  $\mathbb{C}[a, b, \bar{b}]$ .

Let us turn to the problem of computing the intersection of the pair of algebraic surfaces (80,81). From the algebraic geometry theory we know that the result of the intersection of two irreducible surfaces  $S_1$  and  $S_2 \subset \mathbb{CP}^3$  is always a finite collection of curves [18]. An extension of the Bezout's theorem asserts that the product of the degrees of  $S_1$  and  $S_2$  should satisfy the relation,

$$\deg(S_1)\deg(S_2) = \sum_{\alpha} I_{\alpha}(S_1, S_2)C_{\alpha}. \quad (83)$$

where the sum varies over all the irreducible curves  $C_{\alpha}$  of  $S_1 \cap S_2$ . The index  $I_{\alpha}(S_1, S_2)$  is the multiplicity of the intersection of  $S_1$  and  $S_2$  along the component  $C_{\alpha}$ .

The first step of the problem is therefore to find out whether the intersection is irreducible or it degenerates in the union of a given number of curves. This can be answered by exploring a basic fact in commutative algebra that assures us that the ideal generated by the polynomials  $S_1$  and  $S_2$  can be decomposed in terms of more elementary ideal components, see [19] for mathematical details. These components are called primary ideals whose defining polynomials give rise to the varieties corresponding precisely to the irreducible curves coming from the intersection  $S_1 \cap S_2$ . It is fortunate that the existing algorithms for extracting information on this primary decomposition have already been implemented in computer algebra systems such as for example Singular [20]. With the help of this software we find that the ideal associated to the polynomials (80,81) is in fact reducible having three main primary components. Here we are in a favorable situation in which two of them sit exactly at the singular locus of  $S_2$  defined by the lines (82). By analyzing the factorization of the polynomials (80,81) around  $b = 0$  we can easily obtain that the respective intersection indices are  $I_{I_1}(S_1, S_2) = I_{I_2}(S_1, S_2) = 2$ . Considering all these information together with formula (83) we conclude that the third component has to be a spatial curve of degree eight. From the practical point of view such third component is the only non trivial result of the intersection since we are not interested in solutions having null weights.

The next step is to map the surface intersection to a degree eight plane curve by hopefully

determining exactly one of the weights in terms of the curve variables. To this end the result of the primary decomposition for the third component is of any help since the respective basis of the primary ideal is constructed out of a large number of polynomials. This task can however be resolved by means of the following method. We rewrite the the surfaces expressions as univariate polynomials in the variable we want to eliminate which here we choose to be the weight  $\bar{b}$ , namely

$$S_1(\bar{b}) = \mathbf{v}_2 \bar{b}^2 + \mathbf{v}_1 \bar{b} + \mathbf{v}_0, \quad (84)$$

$$S_2(\bar{b}) = \mathbf{u}_4 \bar{b}^4 + \mathbf{u}_3 \bar{b}^3 + \mathbf{u}_2 \bar{b}^2 + \mathbf{u}_1 \bar{b} + \mathbf{u}_0, \quad (85)$$

where the coefficients  $\mathbf{v}_j$  and  $\mathbf{u}_j$  depend on the remaining weights  $a$ ,  $b$  and  $c$ . They are determined by just matching Eqs.(80,81) to Eqs.(84,85) respectively.

We now proceed by making certain linear combinations among  $S_1(\bar{b})$  and  $S_2(\bar{b})$  in order to lower the highest degree in the variable  $\bar{b}$ . For instance this is achieved by defining the following new polynomials,

$$\begin{aligned} \tilde{S}_1(\bar{b}) &= [\mathbf{u}_0 S_1(\bar{b}) - \mathbf{v}_0 S_2(\bar{b})] / \bar{b}, \\ \tilde{S}_2(\bar{b}) &= \mathbf{u}_4 \bar{b}^2 S_1(\bar{b}) - \mathbf{v}_2 S_2(\bar{b}). \end{aligned} \quad (86)$$

Because both  $\tilde{S}_1(\bar{b})$  and  $\tilde{S}_2(\bar{b})$  are algebraic combinations of the starting surfaces they certainly contain the intersection curve that we are searching for. The nice feature of the combination (86) is that the maximal degree of the polynomials has now decreased to three. By repeating this procedure using now in the right hand side of Eq.(86) the new polynomials  $\tilde{S}_1(\bar{b})$  and  $\tilde{S}_2(\bar{b})$  we are able to lower the degree in  $\bar{b}$  once again by one. After three such steps we reach a couple of relations that are both linear in  $\bar{b}$  and this variable can finally be eliminated. By extracting the weight  $\bar{b}$  from one of the relations and substituting the result back into the other we obtain a polynomial in the variables  $a$ ,  $b$  and  $c$  that factorizes in terms of three curves with degrees six, eight and nine. The components with degree six and nine play the role of extraneous factors and they should be discarded since our previous rigorous argument tell us that the sought plane curve should have

degree eight. The explicit expression for this algebraic plane octic curve is,

$$\begin{aligned} C_1(a, b, c) &= (\varepsilon_1 \varepsilon_2 - 1)[a^4 + a^2 b^2 + b^4]^2 + (2 + \varepsilon_1^2 - \varepsilon_1 \varepsilon_2 + \varepsilon_2^2) [\varepsilon_2(a^4 + a^2 b^2 + b^4) + abc^2] abc^2 \\ &- [(\varepsilon_1 \varepsilon_2 - 2)a^4 + (2 - \varepsilon_1 \varepsilon_2 + \varepsilon_2^2)b^4 - \varepsilon_2^2 a^2 b^2 + 2\varepsilon_2 abc^2 + c^4] c^4, \end{aligned} \quad (87)$$

and after a systematic use of the constrain (87) we are able to simplify the expression for the weight  $\bar{b}$  to obtain,

$$\bar{b} = \frac{[(\varepsilon_2 - \varepsilon_1)a^2 + \varepsilon_2 b^2] bc^2 - a[a^4 + a^2 b^2 + b^4 - c^4]}{\varepsilon_2(a^4 + a^2 b^2 + b^4) + 2abc^2}. \quad (88)$$

As possible check of the above results we can substitute the weight (88) in the original polynomial equations defining the surfaces  $S_1$  and  $S_2$ . With the help of a symbolic algebra system one easily find that the relations (80,81) become indeed proportional to the octic plane curve (87).

We now turn our attention to discuss the geometric properties of the degree eight plane curve (87) governing the integrability of the main branch manifold. The basic invariant characterizing a projective algebraic plane curve is the topological genus of the respective compact Riemann surface normalization. In order to compute the genus we need first to identify the singular points and afterwards to investigate their morphology which include the understanding of possible infinitesimal neighboring singularities. By solving the polynomial equations (78) for the plane curve (87) we find a total number of twelve singular points. The first eight of them are located in the affine plane  $c = 1$  and they can be expressed as,

$$P_A = [a_s : b_s : 1], \quad (89)$$

where the values  $a_s$  and  $b_s$  are the non-null solutions of the following relations,

$$\varepsilon_2^2 a_s^3 + (\varepsilon_1 \varepsilon_2 - 2)a_s b_s^2 + \varepsilon_2 b_s = 0, \quad (90)$$

$$\varepsilon_2^2 b_s^3 + (2 - \varepsilon_1 \varepsilon_2 + \varepsilon_2^2)a_s^2 b_s + \varepsilon_2 a_s = 0. \quad (91)$$

The above coupled non-linear equations can be resolved in terms of the roots of a single univariate polynomial with degree eight. The final answer is somewhat cumbersome but for sake of completeness it has been presented in Appendix C. At this point we recall that a relevant index characterizing the morphology a singular point is the multiplicity of the singularity. A singular

point  $[a_0 : b_0 : c_0]$  is said to have multiplicity  $m$  on the plane curve  $C_1(a, b, c)$  if every partial derivative with respect to the coordinates  $a, b$  and  $c$  up to the order  $m - 1$  vanishes at  $[a_0 : b_0 : c_0]$  while at least one of order  $m$  is non null at this point. In addition, the singular point is called ordinary when the factorization of the leading term of the expansion of  $C_1(a, b, c)$  around  $[a_0 : b_0 : c_0]$  gives rise to  $m$  different terms and thus we don't have multiple tangents through this point. It turns out that the Taylor expansion of  $C_1(a, b, 1)$  near to the affine singularities defined by Eqs.(89-91) always produces two distinct tangents for generic values of the parameters  $\varepsilon_1$  and  $\varepsilon_2$ . This means that all the affine singularities are in fact ordinary double singular points.

The remaining four singular points are located at the infinity line  $c = 0$  being just the zeros of the monomial  $C_1(a, b, 0)$ . As a consequence they are independent of  $\varepsilon_1$  and  $\varepsilon_2$  and their explicit coordinates are,

$$P_\infty = [\pm \exp(i\frac{\pi}{3}) : 1 : 0] \quad \text{and} \quad [\pm \exp(i\frac{2\pi}{3}) : 1 : 0]. \quad (92)$$

The singularities at infinity behave as double point with only one tangent but having two branches. They are not ordinary singular points and are usually known in the literature as tac-nodes. Their presence means the existence of extra neighboring singular points and the plane curve  $C_1(a, b, c)$  desingularizes only after the implementation of a sequence of two global birational transformations often named blowing-ups. The main idea of this method is to replace a point in the plane by a projective line which opens more room for the curve to become non-singular in a higher dimensional space. For the technical details concerning this approach in the context of desingularization of an algebraic plane curve we refer to an overview by Abhyankar [21]. As a result the resolution of the singularities of the octic plane curve (87) can be represented by the diagram,

$$\mathbf{C}_1 \xrightarrow{\pi_2} \tilde{\mathbf{C}}_1 \xrightarrow{\pi_1} C_1(a, b, c). \quad (93)$$

where  $\pi_1$  and  $\pi_2$  represent two consecutive blowing-ups, the intermediate curve  $\tilde{\mathbf{C}}_1$  carries infinitely near singularities and the smooth normalization is denoted by  $\mathbf{C}_1$ .

Under the transformation  $\pi_1$  all the affine singular points are mapped onto simple non-singular points while the singularities at the infinity line become four ordinary double points. The latter are called infinitesimal neighboring singular points now sited in  $\tilde{\mathbf{C}}_1$  which are finally resolved with

the help of the second map  $\pi_2$ . The genus of the curve normalization can be computed using the following standard formula valid for plane curves,

$$g(\mathbf{C}_1) = \frac{(D-1)(D-2)}{2} - \sum_P \frac{m_P(m_P-1)}{2}, \quad (94)$$

where  $D$  is the degree of the curve and  $m_P$  denotes the multiplicity of the singular point  $P$ . The sum is taken over all the singular points on the curves that are infinitesimal neighbors of  $C_1(a, b, c)$ .

At this point we know that all the singular points have multiplicity  $m_P = 2$  including those in the infinitesimal neighborhood of the singular points at the infinity line. Considering this information in Eq.(94) we obtain that the genus is,

$$g(\mathbf{C}_1) = \frac{7 \times 6}{2} - \underbrace{12 \times 1}_{\pi_1} - \underbrace{4 \times 1}_{\pi_2} = 5, \quad (95)$$

whose value can be confirmed within symbolic algebra packages capable of computing the genus of plane curves such as Singular [20].

In general, canonical genus five curves are known to be realized as the complete intersection of three quadrics in  $\mathbb{CP}^4$  but they also can degenerate in either hyperelliptic or trigonal curves, see for example [22]. One effective way to shed some light on the actual class of the algebraic curve (87) is to investigate possible mappings to other curves with lower genus. Exploring the fact that the polynomial (87) depends only on even powers of the weight  $c$  we are able to establish the following regular map,

$$\begin{aligned} \mathbf{C}_1(a, b, c) \subset \mathbb{CP}^2 & \xrightarrow{\phi} \mathbf{Q}_1(x, y, z) \subset \mathbb{CP}^2 \\ [a : b : c] & \longmapsto [a^2 : ab : c^2], \end{aligned} \quad (96)$$

where the image of the map  $\phi$  is the algebraic curve  $\mathbf{Q}_1(x, y, z)$  defined by,

$$\begin{aligned} Q_1(x, y, z) &= (\varepsilon_1 \varepsilon_2 - 1)[x^4 + x^2 y^2 + y^4]^2 + (2 + \varepsilon_1^2 - \varepsilon_1 \varepsilon_2 + \varepsilon_2^2)[\varepsilon_2(x^4 + x^2 y^2 + y^4) + x^2 y z] x^2 y z \\ &- [(\varepsilon_1 \varepsilon_2 - 2)x^4 + (2 - \varepsilon_1 \varepsilon_2 + \varepsilon_2^2)y^4 - \varepsilon_2^2 x^2 y^2 + 2\varepsilon_2 x^2 y z + x^2 z^2] x^2 z^2. \end{aligned} \quad (97)$$

The degree of the above map is two because this is the cardinality of the fiber  $\phi^{-1}(P)$  for a generic point  $P \in \mathbf{Q}_1(x, y, z)$ . To make further progress on this admissible double cover we need

to compute the genus of the target curve. In this sense we find that  $Q_1(x, y, z)$  has eight ordinary singular points and one singularity resembling the tacnode behaviour but with higher multiplicity  $m_P = 4$ . For sake of completeness the technical details entering this analysis have been summarized in Appendix C. The curve  $Q_1(x, y, z)$  desingularizes once again after a sequence of two blowing-ups and the desingularization diagram is similar to that shown in (93). Denoting the corresponding blowing-ups by  $\bar{\pi}_1$  and  $\bar{\pi}_2$  we find that genus of this curve is given by,

$$g(\mathbf{Q}_1) = \frac{7 \times 6}{2} - \underbrace{(8 \times 1 + 1 \times 6)}_{\bar{\pi}_1} - \underbrace{1 \times 6}_{\bar{\pi}_2} = 1, \quad (98)$$

where  $\mathbf{Q}_1$  denotes the normalization of  $Q_1(x, y, z)$  which turns out to be an elliptic curve.

Now, putting all these information together we can establish the following commutative diagram,

$$\begin{array}{ccc} \mathbf{C}_1 & \xrightarrow{\psi} & \mathbf{Q}_1 \\ \downarrow \pi_1 \circ \pi_2 & & \downarrow \bar{\pi}_1 \circ \bar{\pi}_2 \\ C_1(a, b, c) & \xrightarrow{\phi} & Q_1(x, y, z). \end{array} \quad (99)$$

where  $\psi$  denotes the morphism induced by the map  $\phi$ .

From the diagram (99) we see that the morphism  $\psi$  has also degree two and therefore the genus five curve  $\mathbf{C}_1$  admits a double covering in terms of the smooth elliptic curve  $\mathbf{Q}_1$ . The existence of such degree two map is known to rule out the possibility that the octic plane curve (87) be sited in the space of either hyperelliptic or trigonal curves [23]. We then conclude that we are in fact dealing with a bielliptic genus five curve whose canonical model is the complete intersection of three independent quadrics in  $\mathbb{CP}^4$ .

We close this subsection emphasizing that the above discussion on the geometric properties of the octic plane curve (87) is valid for generic points on the two-dimensional space generated by the free parameters  $\varepsilon_1$  and  $\varepsilon_2$ . Such family of curves may degenerated to plane curves with lower genus once we restrict  $\varepsilon_1$  and  $\varepsilon_2$  to lie on certain specific submanifolds of the parameter space. This fact can naturally occur when one of the surfaces (80,81) becomes reducible and as result their intersection will give rise to reducible plane curves whose components should have degree lower

than eight. In Appendix B we have in fact remarked that the the surface (81) can degenerate into the product of two cones over plane conics. This happens when the parameters  $\varepsilon_1$  and  $\varepsilon_2$  are constrained to sit in the following one-dimensional submanifolds,

$$\varepsilon_2 \exp(\pm i \frac{\pi}{6}) - \varepsilon_1 \exp(\mp i \frac{\pi}{6}) \mp 2i = 0 \quad (100)$$

$$\varepsilon_2 \exp(\pm i \frac{\pi}{6}) - \varepsilon_1 \exp(\mp i \frac{\pi}{6}) \pm 2i = 0 \quad (101)$$

$$4\varepsilon_1^2 + \varepsilon_1^4 - 2\varepsilon_1^3\varepsilon_2 + 4\varepsilon_2^2 + 3\varepsilon_1^2\varepsilon_2^2 - 2\varepsilon_1\varepsilon_2^3 + \varepsilon_2^4 = 0. \quad (102)$$

It is not difficult to repeat the previous reasoning on the intersection of the surfaces (80,81) now for the particular submanifolds (100-102). Due the factorization of the degree four surface (81) one expects that the result of the intersection will lead us to the product of two quartic algebraic plane curves. This analysis is somehow direct for the first two manifolds (100,101) since one of the parameter can be linearly eliminated but it is more involved for the third non-linear submanifold (102). The technical details entering this analysis are presented in Appendix D and here we only state our main conclusions. From the explicit expressions of these quartic plane curves we conclude that they are singular. The singularities associated to the linear submanifolds (100,101) are single tacnodes at the infinity line while those related to the third submanifold (102) are constituted of two ordinary double points in the affine plane. This means that when the parameters  $\varepsilon_1$  and  $\varepsilon_2$  are constrained in any of the special submanifolds (100-102) the main branch integrable vertex model is therefore governed by plane quartic curves of genus one.

## 5.2 The Special Branch

The situation for the special branch is much simpler since the weight  $\bar{b}$  can be linearly eliminated from the divisor (49). Considering the results of Table (2) one would think that this branch has to be splitted in four different integrable manifolds associated to the four possible values of the parameter  $\tilde{\Lambda}_2 = \pm \exp(\pm i \frac{\pi}{6})$ . However, these apparent distinct vertex models are related to each other and as result we have only one independent integrable manifold. The multiplicative signs are easily gauged away with the help of standard gauge transformations. In addition, the vertex models with  $\tilde{\Lambda}_2 = \exp(i \frac{\pi}{6})$  and  $\tilde{\Lambda}_2 = \exp(-i \frac{\pi}{6})$  can then be connected by applying the Weyl basis



transformation  $1 \leftrightarrow 3$  on the Lax operator (17). Taking this observation into account we find that the expression of the weight  $\bar{b}$  for such single vertex model is,

$$\bar{b} = \frac{\Lambda_0 b c^2}{a^2 + \Lambda_0 b^2}, \quad (103)$$

where we recall that  $\Lambda_0 = \exp(\pm i \frac{\pi}{3})$ .

In order to obtain the constrain among the variables  $a$ ,  $b$  and  $c$  we substitute the above expression for the weight  $\bar{b}$  in the divisor (51). After some simplifications using the identities satisfied by the parameter  $\Lambda_0$  we find the respective algebraic plane curve is given by,

$$C_2(a, b, c) = [a^2 + \Lambda_0 b^2](a^4 + a^2 b^2 + \Lambda_4 a b c^2) + (b^2 - a^2) c^4, \quad (104)$$

where  $\Lambda_4$  is the free parameter.

The sextic plane curve (104) has three singular points being one of them on the affine plane while the others are sitting on the line at infinity. Their explicit coordinates are given by,

$$P_A = [0 : 0 : 1], \quad P_\infty = [\frac{1}{\Lambda_0} : 1 : 0] \text{ and } [-\frac{1}{\Lambda_0} : 1 : 0]. \quad (105)$$

The singularity at the origin of the affine plane is an ordinary double point while the remaining ones behave as tacnodes and the sextic curve is desingularized again by a sequence of two blowing-ups. We can compute the genus of the normalization  $\mathbf{C}_2$  of the sextic curve  $C_2(a, b, c)$  along the same lines presented in previous subsection. By applying the formula (94) we obtain that the genus of  $\mathbf{C}_2$  is,

$$g(\mathbf{C}_2) = \frac{5 \times 4}{2} - \underbrace{3 \times 1}_{\pi_1} - \underbrace{2 \times 1}_{\pi_2} = 5. \quad (106)$$

In what follows we shall argue that the sextic plane curve turns out to be a bielliptic genus five curve as well. The first step is to note that the same two sheeted cover we have discussed before can also be established for the sextic curve (104), namely

$$\begin{aligned} C_2(a, b, c) \subset \mathbb{CP}^2 & \xrightarrow{\phi} Q_2(x, y, z) \subset \mathbb{CP}^2 \\ [a : b : c] & \longmapsto [a^2 : ab : c^2], \end{aligned} \quad (107)$$

where the expression of the algebraic target curve  $Q_2(x, y, z)$  is,

$$Q_2(x, y, z) = [x^2 + \Lambda_0 y^2](x^4 + x^2 y^2 + y^4 + \Lambda_4 x^2 y z) + (y^2 - x^2)x^2 z^2. \quad (108)$$

We next observe that the image curve (108) is quadratic in the variable  $z$  and therefore the linear term on this variable can be eliminated by quadrature. In an analogy to what has been explained in Appendix B this define a birational transformation now in the projective space. More precisely, we are able to put forward the following second mapping,

$$\begin{aligned} Q_2(x, y, z) \subset \mathbb{CP}^2 & \xrightarrow{\tilde{\phi}} \tilde{Q}_2(x_1, y_1, z_1) \subset \mathbb{CP}^2 \\ [x : y : z] & \longmapsto \left[ \tilde{\phi}_1(x, y, z) : \tilde{\phi}_2(x, y, z) : \tilde{\phi}_3(x, y, z) \right], \end{aligned} \quad (109)$$

where the polynomial components of the map (109) are given by,

$$\begin{aligned} \tilde{\phi}_1(x, y, z) &= x(x^2 + \Lambda_0 y^2), \\ \tilde{\phi}_2(x, y, z) &= y(x^2 + \Lambda_0 y^2), \\ \tilde{\phi}_3(x, y, z) &= -i [(\Lambda_4 y(x^2 + \Lambda_0 y^2) + 2(y^2 - x^2)z)], \end{aligned} \quad (110)$$

while the expression of the image curve  $\tilde{Q}_2(x_1, y_1, z_1)$  is,

$$\tilde{Q}_2(x_1, y_1, z_1) = x_1^2 z_1^2 - \frac{4}{\Lambda_0} y_1^4 - (4\Lambda_0 - \Lambda_4^2)x_1^2 y_1^2 + 4x_1^4 \quad (111)$$

Direct inspection of the quartic plane curve (111) reveals us that it can readily be brought into the Jacobi's form of an elliptic curve. Considering the combination of the above two maps we are able to build up the following diagram,

$$\begin{array}{ccc} C_2(a, b, c) & \xrightarrow{\phi} & Q_2(x, y, z) \\ & \searrow & \downarrow \tilde{\phi} \\ & & \tilde{Q}_2(x_1, y_1, z_1) \end{array} \quad (112)$$

The degree of the composition of transformations among varieties having same dimension is the product of the degrees of the individual mappings and from this fact it follows that  $\deg(\tilde{\phi} \circ \phi) = 2$ .

This means that the diagram (112) represents a direct double cover mapping from the singular genus five curve  $C_2(a, b, c)$  to a quartic singular curve with genus one. As before this shows that the sextic plane curve (104) sits in the moduli space of bielliptic curves.

Finally, we remark that another direct consequence of the diagram (112) is that at the special values  $\Lambda_4^2 = 12\Lambda_0, -4\Lambda_0$  the sextic curve (104) should degenerate into a plane curve of genus one. This is because at these parameter values the discriminant of the elliptic curve  $\tilde{Q}_2(x_1, y_1, z_1)$  is zero and as result we have a double cover mapping  $\tilde{\phi} \circ \phi$  from the original plane sextic curve to a rational conic curve. In fact, for  $\Lambda_4^2 = 12\Lambda_0$  and  $-4\Lambda_0$  we verified that sextic curve  $C_2(a, b, c)$  factorizes in terms of the product of two non-singular cubic plane curves.

## 6 The Integrable Lattice Models

We now gathered the basic ingredients to present our main results in terms of the language often used in the modern algebraic theory of exactly solvable lattice systems. At this point we know that the Boltzmann weights of the uncovered vertex models sit on algebraic plane curves and in principle they can be uniformized by means of a single spectral parameter. For future convenience we rewrite the Yang-Baxter algebra in the following more general form,

$$R_{\alpha,\beta}(\lambda, \mu) L_{\alpha,k}(\lambda) L_{\beta,k}(\mu) = L_{\beta,k}(\mu) L_{\alpha,k}(\lambda) R_{\alpha,\beta}(\lambda, \mu), \quad k = 1, \dots, N, \quad (113)$$

where  $\alpha, \beta$  indicate the R-matrix action on the auxiliary spaces while the index  $k$  represents the quantum spaces. The spectral parameters are denoted by  $\lambda$  and  $\mu$ .

We start by presenting the explicit expressions for the Lax operators  $L_{\alpha,k}(\lambda)$  and the R-matrix  $R_{\alpha,\beta}(\lambda, \mu)$ .

## 6.1 Lax operator and R-matrix

Without loss of generality we shall normalize the Lax operator by the weight  $c$ . It turns out that for both branches the Lax operator (17) can be written as follows,

$$\begin{aligned}
L_{\alpha,k}(\lambda) = & a(\lambda)[e_{11}^{(\alpha)} \otimes e_{11}^{(k)} + e_{33}^{(\alpha)} \otimes e_{33}^{(k)}] + b(\lambda)[e_{11}^{(\alpha)} \otimes e_{22}^{(k)} + e_{33}^{(\alpha)} \otimes e_{22}^{(k)}] + \bar{b}(\lambda)[e_{22}^{(\alpha)} \otimes e_{11}^{(k)} + e_{22}^{(\alpha)} \otimes e_{33}^{(k)}] \\
& + [e_{12}^{(\alpha)} \otimes e_{21}^{(k)} + e_{21}^{(\alpha)} \otimes e_{12}^{(k)} + e_{23}^{(\alpha)} \otimes e_{32}^{(k)} + e_{32}^{(\alpha)} \otimes e_{23}^{(k)}] + d(\lambda)[e_{12}^{(\alpha)} \otimes e_{32}^{(k)} + e_{23}^{(\alpha)} \otimes e_{21}^{(k)}] \\
& + \exp(\pm i \frac{\pi}{3})d(\lambda)[e_{21}^{(\alpha)} \otimes e_{23}^{(k)} + e_{32}^{(\alpha)} \otimes e_{12}^{(k)}] + f(\lambda)[e_{11}^{(\alpha)} \otimes e_{33}^{(k)} + e_{33}^{(\alpha)} \otimes e_{11}^{(k)}] + g(\lambda)[e_{22}^{(\alpha)} \otimes e_{22}^{(k)}] \\
& + [a(\lambda) + \exp(\mp i \frac{\pi}{3})f(\lambda)][e_{13}^{(\alpha)} \otimes e_{31}^{(k)}] + [a(\lambda) + \exp(\pm i \frac{\pi}{3})f(\lambda)][e_{31}^{(\alpha)} \otimes e_{13}^{(k)}] \quad (114)
\end{aligned}$$

The structure of the weights  $\bar{b}(\lambda)$ ,  $d(\lambda)$ ,  $f(\lambda)$  and  $g(\lambda)$  are however branch dependent as well as the underlying algebraic plane curves. For the main branch the weight  $\bar{b}(\lambda)$  has been already determined by Eq.(88) while the remaining weights are obtained by solving the divisors (31,38,56). Considering the constrains among the parameters listed in Table (2) we find that the final results for these weights are:

- The main branch

$$\bar{b}(\lambda) = \frac{[(\varepsilon_2 - \varepsilon_1)a^2(\lambda) + \varepsilon_2 b^2(\lambda)]b(\lambda) - [a^4(\lambda) + a^2(\lambda)b^2(\lambda) + b^4(\lambda) - 1]a(\lambda)}{\varepsilon_2[a^4(\lambda) + a^2(\lambda)b^2(\lambda) + b^4(\lambda)] + 2a(\lambda)b(\lambda)}, \quad (115)$$

$$d(\lambda) = \pm \sqrt{\frac{\varepsilon_1 \varepsilon_2 - 1}{\exp(\pm i \frac{\pi}{3})}} \frac{b(\lambda)}{[\varepsilon_1 a(\lambda) + \bar{b}(\lambda)]}, \quad f(\lambda) = \frac{b(\lambda)[\varepsilon_1 a(\lambda)b(\lambda) + b(\lambda)\bar{b}(\lambda) - 1]}{a(\lambda)[\varepsilon_1 a(\lambda) + \bar{b}(\lambda)]}, \quad (116)$$

$$g(\lambda) = \left[ \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1 \varepsilon_2} \right] b(\lambda) + \left[ \frac{(\varepsilon_1 \varepsilon_2 - 1)b(\lambda)}{\varepsilon_1 [\varepsilon_1 a(\lambda) + \bar{b}(\lambda)]} + \frac{\varepsilon_1 a(\lambda) + \bar{b}(\lambda)}{\varepsilon_2 b(\lambda)} \right] \bar{b}(\lambda), \quad (117)$$

where the variables  $a(\lambda)$  and  $b(\lambda)$  fulfill the affine version of the genus five octic plane curve (87),

$$\begin{aligned}
C_2(\lambda) = & (\varepsilon_1 \varepsilon_2 - 1)[a^4(\lambda) + a^2(\lambda)b^2(\lambda) + b^4(\lambda)]^2 - (\varepsilon_1 \varepsilon_2 - 2)a^4(\lambda) + (2 - \varepsilon_1 \varepsilon_2 + \varepsilon_2^2)b^4(\lambda) \\
& + (2 + \varepsilon_1^2 - \varepsilon_1 \varepsilon_2 + \varepsilon_2^2) [\varepsilon_2 \{a^4(\lambda) + a^2(\lambda)b^2(\lambda) + b^4(\lambda)\} + a(\lambda)b(\lambda)] a(\lambda)b(\lambda) \\
& - \varepsilon_2^2 a^2(\lambda)b^2(\lambda) + 2\varepsilon_2 a(\lambda)b(\lambda) + 1. \quad (118)
\end{aligned}$$

Similar results for the special branch are now obtained considering the divisors (44,46,56,103). They can be solved linearly and the expressions for the weights are:

- The special branch

$$\bar{b}(\lambda) = \frac{b(\lambda)}{\exp(\mp i \frac{\pi}{3})a^2(\lambda) + b^2(\lambda)}, \quad d(\lambda) = \frac{b(\lambda)}{a(\lambda)}, \quad (119)$$

$$g(\lambda) = \frac{a^2(\lambda) + \exp(\pm i \frac{2\pi}{3})b^2(\lambda)}{a(\lambda)[a^2(\lambda) + \exp(\pm i \frac{\pi}{3})b^2(\lambda)]}, \quad f(\lambda) = \frac{b^2(\lambda)}{a(\lambda)}, \quad (120)$$

where the variables  $a(\lambda)$  and  $b(\lambda)$  satisfy the affinization of the genus five sextic plane curve (104),

$$C_2(\lambda) = [a^2(\lambda) + \exp(\pm i \frac{\pi}{3})b^2(\lambda)][a^4(\lambda) + a^2(\lambda)b^2(\lambda) + \Lambda_4 a(\lambda)b(\lambda)] + b^2(\lambda) - a^2(\lambda) \quad (121)$$

An important property of the above Lax operators is that there exists a special value for the spectral parameter  $\lambda$  in which they become proportional to the permutator. This turns out to be the point  $\lambda_0$  on both curves (118,121) in which  $a(\lambda_0) = 1$  and  $b(\lambda_0) = 0$ . In fact, considering the expressions for the weights  $\bar{b}(\lambda)$ ,  $d(\lambda)$ ,  $f(\lambda)$  and  $g(\lambda)$  at the value  $\lambda_0$  we obtain<sup>1</sup>,

$$L_{\alpha,k}(\lambda_0) = \mathcal{P}_{\alpha,k} = \sum_{i,j=1}^3 e_{ij}^{(\alpha)} \otimes e_{ji}^{(k)} \quad (122)$$

Let us now turn our attention to the R-matrix. After some cumbersome simplifications we find that the R-matrix has a universal structure for both branches once we write its elements in terms of an enlarged set of weights given by  $a(\lambda)$ ,  $b(\lambda)$ ,  $\bar{b}(\lambda)$ ,  $d(\lambda)$  and  $f(\lambda)$ . The form of the R-matrix becomes similar to that of Lax operators, namely

$$\begin{aligned} R_{\alpha,\beta}(\lambda, \mu) = & \mathbf{a}(\lambda, \mu)[e_{11}^{(\alpha)} \otimes e_{11}^{(\beta)} + e_{33}^{(\alpha)} \otimes e_{33}^{(\beta)}] + \mathbf{b}(\lambda, \mu)[e_{11}^{(\alpha)} \otimes e_{22}^{(\beta)} + e_{33}^{(\alpha)} \otimes e_{22}^{(\beta)}] \\ & + \bar{\mathbf{b}}(\lambda, \mu)[e_{22}^{(\alpha)} \otimes e_{11}^{(\beta)} + e_{22}^{(\alpha)} \otimes e_{33}^{(\beta)}] + [e_{12}^{(\alpha)} \otimes e_{21}^{(\beta)} + e_{21}^{(\alpha)} \otimes e_{12}^{(\beta)} + e_{23}^{(\alpha)} \otimes e_{32}^{(\beta)} + e_{32}^{(\alpha)} \otimes e_{23}^{(\beta)}] \\ & + \mathbf{d}(\lambda, \mu)[e_{12}^{(\alpha)} \otimes e_{32}^{(\beta)} + e_{23}^{(\alpha)} \otimes e_{21}^{(\beta)}] + \exp(\pm i \frac{\pi}{3})\mathbf{d}(\lambda, \mu)[e_{21}^{(\alpha)} \otimes e_{23}^{(\beta)} + e_{32}^{(\alpha)} \otimes e_{12}^{(\beta)}] \\ & + \mathbf{f}(\lambda, \mu)[e_{11}^{(\alpha)} \otimes e_{33}^{(\beta)} + e_{33}^{(\alpha)} \otimes e_{11}^{(\beta)}] + \mathbf{g}(\lambda, \mu)[e_{22}^{(\alpha)} \otimes e_{22}^{(\beta)}] \\ & + [\mathbf{a}(\lambda, \mu) + \exp(\mp i \frac{\pi}{3})\mathbf{f}(\lambda, \mu)][e_{13}^{(\alpha)} \otimes e_{31}^{(\beta)}] + [\mathbf{a}(\lambda, \mu) + \exp(\pm i \frac{\pi}{3})\mathbf{f}(\lambda, \mu)][e_{31}^{(\alpha)} \otimes e_{13}^{(\beta)}] \end{aligned} \quad (123)$$

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<sup>1</sup>Note that for the main branch the direct limit  $\lim_{\lambda \rightarrow \lambda_0} g(\lambda)$  is indefinite. This indeterminacy is evaluated by substituting the expression for  $\bar{b}(\lambda)$  in Eq.(117) and afterwards the high weights powers are reduced with the help of the curve constrain (118). After carrying on these simplifications we find indeed that  $g(\lambda_0) = 1$

where the matrix elements  $\mathbf{a}(\lambda, \mu)$ ,  $\mathbf{b}(\lambda, \mu)$ ,  $\bar{\mathbf{b}}(\lambda, \mu)$ ,  $\mathbf{d}(\lambda, \mu)$ ,  $\mathbf{f}(\lambda, \mu)$ , and  $\mathbf{g}(\lambda, \mu)$  are given by,

$$\mathbf{a}(\lambda, \mu) = \frac{a(\lambda) + [\bar{b}(\lambda)a(\mu) - a(\lambda)\bar{b}(\mu)]b(\mu)}{a(\mu)}, \quad (124)$$

$$\mathbf{b}(\lambda, \mu) = \frac{[1 - b(\mu)\bar{b}(\mu)]a(\lambda)b(\lambda) - [1 - b(\lambda)\bar{b}(\lambda)]a(\mu)b(\mu)}{a(\lambda)a(\mu)}, \quad (125)$$

$$\bar{\mathbf{b}}(\lambda, \mu) = \bar{b}(\lambda)a(\mu) - a(\lambda)\bar{b}(\mu), \quad (126)$$

$$\mathbf{d}(\lambda, \mu) = \frac{d(\lambda)a(\mu)[\bar{b}(\lambda)a(\mu) - a(\lambda)\bar{b}(\mu)]}{f(\lambda)a(\mu)b(\mu) + \bar{b}(\lambda)[1 - b(\mu)\bar{b}(\mu)]}, \quad (127)$$

$$\mathbf{f}(\lambda, \mu) = \frac{a(\mu)[\bar{b}(\lambda)a(\mu) - a(\lambda)\bar{b}(\mu)][f(\lambda)a(\mu) - \bar{b}(\lambda)\bar{b}(\mu)]}{f(\lambda)a(\mu)b(\mu) + \bar{b}(\lambda)[1 - b(\mu)\bar{b}(\mu)]}, \quad (128)$$

$$\mathbf{g}(\lambda, \mu) = -\mathbf{d}(\lambda, \mu) \frac{[\mathbf{f}(\mu, \lambda) + \mathbf{a}(\mu, \lambda)]}{\mathbf{d}(\mu, \lambda)}. \quad (129)$$

By construction this R-matrix satisfies the unitarity property which now can be stated as,

$$R_{\alpha,\beta}(\lambda, \mu)R_{\beta,\alpha}(\mu, \lambda) = a(\lambda, \mu)a(\mu, \lambda)I_3 \otimes I_3, \quad (130)$$

being also a regular matrix at the point  $\lambda_0$ ,  $R_{\alpha,\beta}(\lambda_0, \lambda_0) = \mathcal{P}_{\alpha,\beta}$ .

In addition to that, we have verified by means of the algebraic procedure explained in Appendix A that the R-matrix satisfies the famous Yang-Baxter equation,

$$R_{12}(\lambda_1, \lambda_2)R_{13}(\lambda_1, \lambda_3)R_{23}(\lambda_2, \lambda_3) = R_{23}(\lambda_2, \lambda_3)R_{13}(\lambda_1, \lambda_3)R_{12}(\lambda_1, \lambda_2) \quad (131)$$

which is sufficient condition for the associativity of the Yang-Baxter algebra (113).

We finally remark that given a solution of the Yang-Baxter equation we can easily produce other equivalent multiparametric solutions by means of the so-called twist transformations. These are natural symmetries underlying the Yang-Baxter algebra and for a more detailed discussion see for example references [24]. It turns out that a special type of twist will be helpful to make a correspondence between our respective quantum spin-1 chains and that derived in the work by

Alcaraz and Bariev [12]. We find that in our case the most general diagonal twist that is compatible with integrability has the following form,

$$\mathcal{G}(\tau_1, \tau_2, \tau_3) = \text{diag}(1, \tau_1, \tau_1^2 | \tau_2, \tau_3, \frac{\tau_3^2}{\tau_2} | \tau_2, \frac{\tau_3^2}{\tau_1}, \frac{\tau_3^4}{\tau_1^2 \tau_2^2}) \quad (132)$$

where  $\tau_1$ ,  $\tau_2$  and  $\tau_3$  are free additional parameters.

It can be checked easily that the following transformed Lax operator and R-matrix

$$\mathcal{G}_{k,\alpha}(\tau_1, \tau_2, \tau_3) L_{\alpha,k}(\lambda) \mathcal{G}_{\alpha,k}^{-1}(\tau_1, \tau_2, \tau_3) \quad \text{and} \quad \mathcal{G}_{\beta,\alpha}(\tau_1, \tau_2, \tau_3) R_{\alpha,\beta}(\lambda, \mu) \mathcal{G}_{\alpha,\beta}^{-1}(\tau_1, \tau_2, \tau_3) \quad (133)$$

is still another solution for the Yang-Baxter algebra(113).

## 6.2 Spin Chain Hamiltonians

The expansion of the logarithm of the transfer matrix  $T(\lambda)$  around the regular point  $\lambda_0$  is known to produce a set of mutually commuting operators. Of particular interest is the Hamiltonian describing the interaction of nearest neighbors spins variables on the lattice,

$$H = \frac{\partial}{\partial \lambda} \ln T(\lambda)|_{\lambda=\lambda_0} = \sum_{k=1}^N H_{k,k+1} \quad (134)$$

where  $H_{k,k+1} = \mathcal{P}_{k,k+1} \frac{\partial}{\partial \lambda} L_{k,k+1}(\lambda)|_{\lambda=\lambda_0}$  and boundary periodic conditions  $H_{N,N+1} = H_{N,1}$  are assumed.

In order to compute the Hamiltonian we just need to take the derivative on the Lax operators at the point  $\lambda = \lambda_0$  and impose that  $a(\lambda_0) = 1$  and  $b(\lambda_0) = 0$ . The derivatives of the weights  $a(\lambda)$  and  $b(\lambda)$  are then related under derivation of the algebraic curves (118,121) and evaluating the results at  $\lambda = \lambda_0$ . In what follows we shall list the final expressions for the two-body Hamiltonians in the Weyl basis:

- The main branch

The spin chain for the main branch is given by,

$$\begin{aligned}
H_{k,k+1}^{(\pm)}(\varepsilon_1, \varepsilon_2) = & J_1 [e_{11}^{(k)} e_{11}^{(k+1)} + e_{33}^{(k)} e_{33}^{(k+1)}] - [e_{21}^{(k)} e_{12}^{(k+1)} + e_{23}^{(k)} e_{32}^{(k+1)}] - \frac{\varepsilon_2}{\varepsilon_1} [e_{12}^{(k)} e_{21}^{(k+1)} + e_{32}^{(k)} e_{23}^{(k+1)}] \\
& + \left[ \frac{\exp(\pm i \frac{\pi}{3})}{\varepsilon_1} + J_1 \right] e_{11}^{(k)} e_{33}^{(k+1)} + \left[ \frac{\exp(\mp i \frac{\pi}{3})}{\varepsilon_1} + J_1 \right] e_{33}^{(k)} e_{11}^{(k+1)} + \frac{1}{\varepsilon_1} [e_{13}^{(k)} e_{31}^{(k+1)} + e_{31}^{(k)} e_{13}^{(k+1)}] \\
& + J_2 \exp(\pm i \frac{\pi}{6}) [e_{12}^{(k)} e_{32}^{(k+1)} + e_{21}^{(k)} e_{23}^{(k+1)}] + J_2 \exp(\mp i \frac{\pi}{6}) [e_{23}^{(k)} e_{21}^{(k+1)} + e_{32}^{(k)} e_{12}^{(k+1)}] \\
& - \left[ \frac{1}{\varepsilon_1} + J_1 \right] e_{22}^{(k)} e_{22}^{(k+1)}
\end{aligned} \tag{135}$$

where the dependence of the couplings  $J_1$  and  $J_2$  on the free parameters  $\varepsilon_1$  and  $\varepsilon_2$  are,

$$J_1 = \frac{\varepsilon_1^2 - \varepsilon_1 \varepsilon_2 + \varepsilon_2^2}{4\varepsilon_1} \quad \text{and} \quad J_2 = \frac{\sqrt{\varepsilon_1 \varepsilon_2 - 1}}{\varepsilon_1}. \tag{136}$$

We now remark that the one-parameter integrable spin-1 chain found previously by Alcaraz and Bariev [12] is in fact a particular case of our Hamiltonian (135) when the respective parameters sit on the subspace  $\varepsilon_1 = \pm \varepsilon_2$ . Let us denote the two-body Hamiltonian derived in the work [12] by  $\bar{H}_{k,k+1}(t_p, \epsilon)$  where  $\epsilon = \pm$  and  $t_p$  is the single free parameter in the notation of this reference<sup>2</sup>. With help of a special case of the twisted transformation (132) we have been able to verify the following correspondence among two-body operators,

$$\begin{aligned}
-\bar{H}_{k,k+1}(t_p, \epsilon) = & \frac{1}{\sqrt{\epsilon}} \mathcal{G}_{k+1,k}(1, \frac{1}{\sqrt{\epsilon}}, \epsilon) H_{k,k+1}^{(+)} \left( -\frac{\sqrt{\epsilon}}{t_p}, -\frac{1}{\sqrt{\epsilon t_p}} \right) \mathcal{G}_{k+1,k}^{-1}(1, \frac{1}{\sqrt{\epsilon}}, \epsilon) + \frac{(\epsilon - 2)}{4t_p} [S_k^z + S_{k+1}^z] \\
& + \frac{i\sqrt{3}t_p}{4\epsilon} [S_k^z - S_{k+1}^z] - \frac{i\sqrt{3}t_p}{4\epsilon} [(S_k^z)^2 - (S_{k+1}^z)^2]
\end{aligned} \tag{137}$$

where  $S_k^z = e_{11}^{(k)} - e_{33}^{(k)}$  denotes the azimuthal component of the spin-1 operator. The second term in (137) is proportional to the azimuthal magnetic field and can always be added since it commutes with Hamiltonian while the last two terms vanish under periodic boundary condition and do not contribute to the volume Hamiltonian. We note from Eq.(137) that the twist was only necessary to fit the case  $\epsilon = -1$ .

Of course one can use the more general twist transformation (132) to generate a family of exactly solvable multiparametric Hamiltonians. Because this is a diagonal twist it does not spoil

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<sup>2</sup>F.C. Alcaraz informed us that the coupling  $u$  in the work [12] should be read as  $u = \frac{\epsilon t_p}{2} + \frac{(2-\epsilon)}{2t_p}$ .



the  $U(1)$  symmetry and the diagonalization of the respective vertex model transfer matrix could in principle be tackled by general algebraic framework proposed in [10].

- The Special branch

In this case we have one-parameter spin chain Hamiltonian and the expression for the corresponding two-body operator is,

$$\begin{aligned} H_{k,k+1}^{(\pm)}(\Lambda_4) &= \frac{\Lambda_4}{4} [e_{11}^{(k)} e_{11}^{(k+1)} + e_{33}^{(k)} e_{33}^{(k+1)} + e_{11}^{(k)} e_{33}^{(k+1)} + e_{33}^{(k)} e_{11}^{(k+1)} - e_{22}^{(k)} e_{22}^{(k+1)}] - [e_{23}^{(k)} e_{21}^{(k+1)} + e_{32}^{(k)} e_{12}^{(k+1)}] \\ &- [e_{21}^{(k)} e_{12}^{(k+1)} + e_{23}^{(k)} e_{32}^{(k+1)}] - \exp(\pm i \frac{\pi}{3}) [e_{12}^{(k)} e_{32}^{(k+1)} + e_{21}^{(k)} e_{23}^{(k+1)} + e_{12}^{(k)} e_{21}^{(k+1)} + e_{32}^{(k)} e_{23}^{(k+1)}] \end{aligned} \quad (138)$$

## 7 Conclusions

In this paper we have investigated the Yang-Baxter algebra for three-state vertex model whose statistical configurations are invariant by the  $U(1)$  invariance but break in an explicit way the parity-time reversal symmetry. We argued that the assumption of unitarity of the respective R-matrix imposes us that the functional equations derived from the Yang-Baxter algebra are anti-symmetrical on the exchange of the Boltzmann weights of distinct Lax operators. This property provides us the means to disentangle involved high degree functional relations in a rather systematic way. The integrable manifolds are found by intersecting a number of prime divisors associated to polynomial equations which are naturally separable on the distinct weights labels. We have been able to uncover two families of integrable nineteen vertex models whose weights are lying on bielliptic algebraic curves of genus five. For the family having two free parameters this comes about after dealing with the problem of the intersection of two projective surfaces: one of them a rational cubic surface and the other a cone generated by an elliptic curve. We have pointed out that genus five bielliptic curves can generate to standard elliptic curves when the respective free parameters are restricted to particular subspaces.

The Lax operators have a regular point in which they become proportional to the permutator and the respective two families of exactly solvable quantum spin-1 chains have been computed. We have found that our two-parameter Hamiltonian family generalizes the integrable one-parameter

spin-1 chain discovered by Alcaraz and Bariev [12]. We exhibit a relationship between these Hamiltonians when our free parameters are restricted to the subspace  $\varepsilon_1 = \pm\varepsilon_2$ . We have found that the R-matrix has the same general form for both family of vertex models provided we write it as function of a suitable subset of Boltzmann weights. The R-matrix is non-additive with respect to the spectral parameters and we have verified that it satisfies the Yang-Baxter equation by means of computer algebra system.

A natural question to be asked is whether these integrable vertex models admit an adequate description in the framework of quantum groups such as turned out to be the case of chiral Potts model [25]. We think that a possible hint into this direction comes from the degeneration of the octic plane curve (87) into a rational curve for specific values of the parameters  $\varepsilon_1$  and  $\varepsilon_2$  discussed in Appendix D. This fact has motivated us to search for a relation among the main branch Hamiltonian at such particular parameter values and known rational quantum spin-1 chains. To this end we have been able to relate the operator  $H^{(\pm)}(\varepsilon_1 = 2, \varepsilon_2 = 2)$  defined by Eq.(135) to that of the trigonometric spin-1 chain based on the quantum superalgebra  $U_{\bar{q}}[\text{Osp}(1|2)]$ , often referred as the Izergin-Korepin model [26], when the deformation parameter is  $\bar{q} = \exp(\mp i\frac{\pi}{3})$ . This suggests that the vertex models obtained in this paper may be also originated from some non-generic three-dimensional representation of the  $U_{\bar{q}}[\text{Osp}(1|2)]$  superalgebra probably at roots of unity. The immediate difficult would be to find the appropriate representation that is able to reproduce the pertinent complete intersection the three quadrics by means of the quantum group machinery. Hopefully, this observation will prompt further investigations on other mathematical properties that are hidden in these vertex models.

Finally, we expect that the approach used in this paper to solve a number of entangled functional equations could also be applied to study the Yang-Baxter algebra associated to generalized nineteen vertex models or even high-state vertex models. We plan to investigate some of these problems in future works.

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## Appendix A: Extra Functional Relations

In this appendix we present the remaining functional relations coming from the Yang-Baxter algebra (5) not presented in the main text. Besides the relations having four terms explicitly exhibited in section 4.3 we have the following extra twelve equations,

$$\mathbf{a}\bar{h}'a'' - \bar{\mathbf{d}}c'\bar{d}'' - \mathbf{f}\bar{h}'f'' - \bar{\mathbf{h}}a'h'' = 0, \quad (\text{A.1})$$

$$\mathbf{h}f'a'' - \mathbf{d}c'\bar{d}'' - \mathbf{h}a'f'' - \mathbf{f}h'\bar{h}'' = 0, \quad (\text{A.2})$$

$$\bar{\mathbf{h}}f'a'' - \bar{\mathbf{d}}c'd'' - \bar{\mathbf{h}}a'f'' - \mathbf{f}\bar{h}'h'' = 0, \quad (\text{A.3})$$

$$\mathbf{d}c'\bar{d}'' + \mathbf{h}\bar{h}'f'' + \mathbf{f}a'h'' - \mathbf{a}f'\bar{h}'' = 0, \quad (\text{A.4})$$

$$\bar{\mathbf{d}}c'd'' + \bar{\mathbf{h}}h'f'' + \mathbf{f}a'h'' - \mathbf{a}f'h'' = 0, \quad (\text{A.5})$$

$$\mathbf{d}c'd'' + \mathbf{h}\bar{h}'h'' - \bar{\mathbf{h}}h'\bar{h}'' - \bar{\mathbf{d}}c'\bar{d}'' = 0, \quad (\text{A.6})$$

$$\mathbf{a}d'c'' - \mathbf{g}c'd'' - \bar{\mathbf{d}}h'f'' - \mathbf{d}a'h'' = 0, \quad (\text{A.7})$$

$$\mathbf{a}\bar{d}'c'' - \mathbf{g}c'\bar{d}'' - \mathbf{d}\bar{h}'f'' - \bar{\mathbf{d}}a'h'' = 0, \quad (\text{A.8})$$

$$\mathbf{c}d'a'' - \mathbf{h}a'd'' - \mathbf{f}h'\bar{d}'' - \mathbf{d}c'g'' = 0, \quad (\text{A.9})$$

$$\mathbf{c}\bar{d}'a'' - \bar{\mathbf{h}}a'\bar{d}'' - \mathbf{f}\bar{h}'d'' - \bar{\mathbf{d}}c'g'' = 0, \quad (\text{A.10})$$

$$\mathbf{c}f'c'' - \mathbf{d}g'\bar{d}'' - \mathbf{h}c'f'' - \mathbf{f}c'\bar{h}'' = 0, \quad (\text{A.11})$$

$$\mathbf{c}f'c'' - \bar{\mathbf{d}}g'd'' - \bar{\mathbf{h}}c'f'' - \mathbf{f}c'h'' = 0. \quad (\text{A.12})$$

In addition to that we have also functional relations involving five terms. Contrary to what happen so far their total number remains unchanged after the solution of the two terms functional

relations. Following Table (2) we have twelve functional equations which are given by,

$$\mathbf{b}\bar{c}'\bar{b}'' + \mathbf{c}g'c'' - \mathbf{d}\bar{h}'d'' - \bar{\mathbf{d}}a'd'' - \mathbf{g}\bar{c}'g'' = 0, \quad (\text{A.13})$$

$$\bar{\mathbf{d}}\bar{b}'\bar{b}'' + \bar{\mathbf{h}}d'c'' - \mathbf{g}\bar{c}'d'' - \bar{\mathbf{d}}a'f'' - \mathbf{d}\bar{h}'h'' = 0, \quad (\text{A.14})$$

$$\mathbf{h}\bar{h}'d'' + \mathbf{f}a'd'' - \mathbf{b}b'd'' + \mathbf{d}\bar{c}'g'' - \mathbf{c}d'h'' = 0, \quad (\text{A.15})$$

$$\mathbf{d}\bar{b}'\bar{b}'' + \mathbf{h}\bar{d}'c'' - \mathbf{g}\bar{c}'d'' - \mathbf{d}a'f'' - \bar{\mathbf{d}}h'h'' = 0, \quad (\text{A.16})$$

$$\bar{\mathbf{b}}c'b'' + \mathbf{c}\bar{h}'c'' - \bar{\mathbf{d}}g'd'' - \mathbf{f}c'f'' - \bar{\mathbf{h}}c'h'' = 0, \quad (\text{A.17})$$

$$\bar{\mathbf{b}}\bar{b}'d'' - \mathbf{g}g'd'' - \mathbf{d}\bar{c}'f'' + \mathbf{c}\bar{d}'g'' - \bar{\mathbf{d}}c'h'' = 0, \quad (\text{A.18})$$

$$\bar{\mathbf{d}}b'b'' + \mathbf{g}\bar{d}'c'' - \mathbf{f}c'd'' - \bar{\mathbf{h}}c'd'' - \bar{\mathbf{d}}g'g'' = 0, \quad (\text{A.19})$$

$$\mathbf{f}a'd'' - \mathbf{b}b'd'' + \bar{\mathbf{h}}h'd'' + \bar{\mathbf{d}}c'g'' - \mathbf{c}\bar{d}'h'' = 0, \quad (\text{A.20})$$

$$\mathbf{b}\bar{c}'d'' + \mathbf{c}g'c'' - \mathbf{d}a'd'' - \bar{\mathbf{d}}h'd'' - \mathbf{g}\bar{c}'g'' = 0, \quad (\text{A.21})$$

$$\bar{\mathbf{b}}\bar{b}'d'' - \mathbf{g}g'd'' - \bar{\mathbf{d}}c'f'' + \mathbf{c}\bar{d}'g'' - \mathbf{d}c'h'' = 0, \quad (\text{A.22})$$

$$\mathbf{g}d'c'' - \mathbf{h}c'd'' - \mathbf{f}c'd'' - \mathbf{d}b'b'' + \mathbf{d}g'g'' = 0, \quad (\text{A.23})$$

$$\bar{\mathbf{b}}c'b'' + \mathbf{c}h'c'' - \mathbf{d}g'd'' - \mathbf{f}c'f'' - \mathbf{h}c'h'' = 0. \quad (\text{A.24})$$

An effective way to check that all the above equations are indeed satisfied for the main branch is to proceed as follows. After extracting linearly the weights  $d$ ,  $f$ ,  $g$ ,  $h$  and  $\bar{h}$  from the divisors (31,38,56,69) the functional equations become polynomials only in the variables  $a$ ,  $b$ ,  $\bar{b}$  and  $c$  for both indices labels. In addition to that these weights are constrained by the remaining divisors (41,43). The main idea of our procedure is to replace in a given functional equation powers of a subset of variables with the help of the last two divisors:

- Step One

We have already mentioned that the functional equations depend only on even powers of the weight  $c$ . The power  $c^2$  can easily be extracted from the divisor (41) or equivalently from the surface (80). Denoting this amplitude by  $\text{auxc}$  we obtain,

$$\text{auxc} = \frac{ab^2 + \varepsilon_1 a^2 \bar{b} + \varepsilon_2 b^2 \bar{b} + a\bar{b}^2}{\varepsilon_2 b} \quad (\text{A.25})$$

Now we inspect the highest power in a given polynomial equation denoted here generically by  $\text{eq}[*]$ . Assuming that this power is for example six the dependence on the weight  $c$  can be

systematically replaced using the following Mathematica code,

$$\text{eq1} = \text{Factor}[\text{eq}[*]] \quad (\text{A.26})$$

$$\text{eq2} = \text{Factor}[\text{eq1} /. \{[c']^6 \rightarrow \text{auxc}' [c']^4, [c'']^6 \rightarrow \text{auxc}'' [c'']^4\}] \quad (\text{A.27})$$

$$\text{eq3} = \text{Factor}[\text{eq2} /. \{[c']^4 \rightarrow \text{auxc}' [c']^2, [c'']^4 \rightarrow \text{auxc}'' [c'']^2\}] \quad (\text{A.28})$$

$$\text{eq4} = \text{Factor}[\text{eq3} /. \{[c']^2 \rightarrow \text{auxc}', [c'']^2 \rightarrow \text{auxc}''\}], \quad (\text{A.29})$$

where  $\text{auxc}'$  and  $\text{auxc}''$  are given by Eq.(A.25) with weights labeled by  $'$  and  $''$ , respectively.

• Step Two

We next use the same method to eliminate other underisable powers now with the help of the divisor (43). For example, we can use this divisor to eliminate the terms that contain powers higher or equal to four in the weight  $\bar{b}$ . Denoting the quartic power on  $\bar{b}$  by  $\text{aux}\bar{b}$  we find that its expression from surface (81) is,

$$\text{aux}\bar{b} = \varepsilon_2^2 b^2 [\varepsilon_2 a \bar{b} + a^2 - \varepsilon_1 a \bar{b} - \bar{b}^2] + \varepsilon_1 \varepsilon_2 b^2 [b^2 + \bar{b}(\varepsilon_1 a + \bar{b})] - [b^2 + \varepsilon_1 a \bar{b}][b^2 + \bar{b}(\varepsilon_1 a + 2\bar{b})] \quad (\text{A.30})$$

Considering that highest power in the resulting polynomial eq4 on the weight  $\bar{b}$  is seven the underisable terms can be replaced as follows,

$$\text{eq5} = \text{Factor}[\text{eq4} /. \{[\bar{b}']^7 \rightarrow \text{aux}\bar{b}' [\bar{b}']^3, [\bar{b}'']^7 \rightarrow \text{aux}\bar{b}'' [\bar{b}'']^3\}] \quad (\text{A.31})$$

$$\text{eq6} = \text{Factor}[\text{eq5} /. \{[\bar{b}']^6 \rightarrow \text{aux}\bar{b}' [\bar{b}']^2, [\bar{b}'']^6 \rightarrow \text{aux}\bar{b}'' [\bar{b}'']^2\}] \quad (\text{A.32})$$

$$\text{eq7} = \text{Factor}[\text{eq6} /. \{[\bar{b}']^5 \rightarrow \text{aux}\bar{b}' [\bar{b}'], [\bar{b}'']^5 \rightarrow \text{aux}\bar{b}'' [\bar{b}'']\}] \quad (\text{A.33})$$

$$\text{eq8} = \text{Factor}[\text{eq7} /. \{[\bar{b}']^4 \rightarrow \text{aux}\bar{b}', [\bar{b}'']^4 \rightarrow \text{aux}\bar{b}''\}], \quad (\text{A.34})$$

where  $\text{aux}\bar{b}'$  and  $\text{aux}\bar{b}''$  are obtained from (A.30) by using the respective label on the weights.

• Step Three

It turns out that the final polynomial relation eq8 is either automatically zero or becomes proportional to the factor  $\Lambda_0^2 - \Lambda_0 + 1$ . In the latter case we can use the simple substitution,

$$\text{eqend} = \text{Factor}[\text{eq8} /. \Lambda_0^2 \rightarrow \Lambda_0 - 1] \quad (\text{A.35})$$

We finally remark that similar verification for the special branch is much simpler since the weight  $\bar{b}$  can be easily extracted from the divisor (49). In this case the functional relations become dependent only on the weights  $a$ ,  $b$  and  $c$ . We now use the algebraic plane curve (104) to extract the power  $c^4$  and denoting it by  $\text{buxc}$  we obtain,

$$\text{buxc} = \frac{a^6 + a^4b^2 + \Lambda_0 a^4b^2 + a^2b^4 + \Lambda_0 a^2b^4 + \Lambda_0 b^6 + \Lambda_4 a^3bc^2 + \Lambda_0 \Lambda_4 ab^3c^2}{a^2 - b^2} \quad (\text{A.36})$$

We find that the highest power on the weight  $c$  is always governed by  $c^4$  and thus a given polynomial  $\text{eq}[*]$  can be verified through the steps,

$$\text{eq1} = \text{Factor}[\text{eq}[*]] \quad (\text{A.37})$$

$$\text{eq2} = \text{Factor}[\text{eq1} /. \{[c']^4 \rightarrow \text{buxc}', [c'']^4 \rightarrow \text{buxc}''\}] \quad (\text{A.38})$$

$$\text{eqend} = \text{Factor}[\text{eq2} /. \Lambda_0^2 \rightarrow \Lambda_0 - 1] \quad (\text{A.39})$$

where  $\text{buxc}'$  and  $\text{buxc}''$  are determined in terms of Eq.(A.36).

## Appendix B: Elliptic Curves

In this appendix we shall show that the cone (81) is in fact defined over an elliptic curve. To this end it is sufficient to work in a given affine chart and here we choose the one defined by setting  $\bar{b} = 1$ . In this affine chart we can re-scale the coordinates as follows,

$$a = x\bar{b} \quad \text{and} \quad b = y\bar{b}, \quad (\text{B.1})$$

and by substituting this re-scaling of coordinates in Eq.(81) we find that the polynomial  $S_2(x\bar{b}, y\bar{b}, \bar{b}) \subset \mathbb{C}[x, y]$  becomes,

$$\begin{aligned} S_2(x, y) &= (\varepsilon_1^2 - \varepsilon_2^2 y^2) x^2 + [2\varepsilon_1 - (\varepsilon_1^2 \varepsilon_2 + \varepsilon_2^3 - 2\varepsilon_1 - \varepsilon_1 \varepsilon_2^2) y^2] x \\ &+ 1 + (2 - \varepsilon_1 \varepsilon_2 + \varepsilon_2^2) y^2 + (1 - \varepsilon_1 \varepsilon_2) y^4. \end{aligned} \quad (\text{B.2})$$

We now can complete the square on the variable  $x$  making it possible the elimination of the linear term on  $x$  of the polynomial (B.2). This is done at the expense of adding an extra factor

that depends only on the variable  $y$  and together with the last term of Eq.(B.2) results in a polynomial that factorizes into two pieces. As a result we are able to define the following one-to-one transformation,

$$\begin{aligned} x &\longmapsto \frac{R_1(y_1)x_1 - R_2(y_1)}{R_3(y_1)}, \\ y &\longmapsto y_1 \end{aligned} \quad (\text{B.3})$$

where the expressions of the polynomials  $R_1(y_1)$ ,  $R_2(y_1)$  and  $R_3(y_1)$  are,

$$\begin{aligned} R_1(y_1) &= 4i\varepsilon_2\sqrt{1-\varepsilon_1\varepsilon_2}(\varepsilon_1^2 - \varepsilon_2^2 y_1^2)y_1, \\ R_2(y_1) &= 2i(\varepsilon_1^2 - \varepsilon_2^2 y_1^2) [2\varepsilon_1 - (\varepsilon_1^2\varepsilon_2 + \varepsilon_2^3 - 2\varepsilon_1 - \varepsilon_1\varepsilon_2^2)y_1^2], \\ R_3(y_1) &= 4i(\varepsilon_1^2 - \varepsilon_2^2 y_1^2)^2. \end{aligned} \quad (\text{B.4})$$

The map (B.3,B.4) is known in the literature as de Jonquières transformation and the zero set of  $S_2(x, y)$  in the new variables  $x_1$  and  $y_1$  turns out to be equivalent to the curve,

$$x_1^2 = y_1^4 + \frac{[8 + \varepsilon_1^4 - 2\varepsilon_1^3\varepsilon_2 + 4\varepsilon_2^2 + \varepsilon_2^4 - 2\varepsilon_1\varepsilon_2(4 + \varepsilon_2^2) + \varepsilon_1^2(4 + 3\varepsilon_2^2)]}{4(1 - \varepsilon_1\varepsilon_2)}y_1^2 + 1. \quad (\text{B.5})$$

The above polynomial has already the form of an elliptic curve since the right-hand side of Eq.(B.5) is a degree four polynomial in  $\mathbb{C}[y_1]$ . In fact, this curve can easily be brought into the form of a Jacobi quartic. Let  $\pm r_1$  and  $\pm r_2$  be the roots of the biquadratic polynomial on the variable  $y_1$ . Then by means of the straightforward re-scaling of coordinates,  $x_1 = r_1 r_2 x_2$  and  $y_1 = r_1 y_2$  the plane curve (B.5) can be rewritten as

$$x_2^2 = (1 - y_2^2)(1 - k^2 y_2^2), \quad (\text{B.6})$$

whose corresponding modulus parameter is  $k = \frac{r_1}{r_2}$ .

We remark that all the above reasoning is valid as long as the discriminant of corresponding biquadratic polynomial on the variable  $y_1$  is not zero. By direct inspection of the right-hand of Eq.(B.5) one finds that the expression of such discriminant is,

$$\Delta = [8 + \varepsilon_1^4 - 2\varepsilon_1^3\varepsilon_2 + 4\varepsilon_2^2 + \varepsilon_2^4 - 2\varepsilon_1\varepsilon_2(4 + \varepsilon_2^2) + \varepsilon_1^2(4 + 3\varepsilon_2^2)]^2 - 64(1 - \varepsilon_1\varepsilon_2)^2, \quad (\text{B.7})$$

It turns out that when  $\Delta = 0$  the elliptic plane curve (B.5) can be factorized in terms of two conics and therefore the original surface (81) becomes rational ruled. From Eq.(B.7) it is not

difficult to find this generation occurs in the following one-dimensional submanifolds,

$$4 - 2\varepsilon_1 + \varepsilon_1^2 - 2\varepsilon_2 - \varepsilon_1\varepsilon_2 + \varepsilon_2^2 = 0, \quad (\text{B.8})$$

$$4 + 2\varepsilon_1 + \varepsilon_1^2 + 2\varepsilon_2 - \varepsilon_1\varepsilon_2 + \varepsilon_2^2 = 0, \quad (\text{B.9})$$

$$4\varepsilon_1^2 + \varepsilon_1^4 - 2\varepsilon_1^3\varepsilon_2 + 4\varepsilon_2^2 + 3\varepsilon_1^2\varepsilon_2^2 - 2\varepsilon_1\varepsilon_2^3 + \varepsilon_2^4 = 0. \quad (\text{B.10})$$

Note that the above constraints are symmetrical under the exchange of parameters  $\varepsilon_1 \leftrightarrow \varepsilon_2$  emphasizing that our initial choice of free parameters was indeed appropriate. In addition, we observe that the first two submanifolds (B.8,B.9) can be further reduced as the product of linear terms given by Eq.(100,101), respectively.

## Appendix C: Singularities of Curves

The purpose of this section is to present the technical details concerning the singular locus of the degree eight algebraic curves discussed in subsection 5.1:

- The Curve  $C_1(a, b, c)$

We start by recalling that the singularities of this curve are sited in the following points,

$$[a_s : b_s : 1], \quad [\pm \exp(i\frac{\pi}{3}) : 1 : 0], \quad [\pm \exp(i\frac{2\pi}{3}) : 1 : 0], \quad (\text{C.1})$$

where the coordinates  $a_s$  and  $b_s$  are a subset of solutions of the relations,

$$\varepsilon_2^2 a_s^3 + (\varepsilon_1 \varepsilon_2 - 2) a_s b_s^2 + \varepsilon_2 b_s = 0, \quad (\text{C.2})$$

$$\varepsilon_2^2 b_s^3 + (2 - \varepsilon_1 \varepsilon_2 + \varepsilon_2^2) a_s^2 b_s + \varepsilon_2 a_s = 0. \quad (\text{C.3})$$

The above equations can be solved by first considering the resultant of the polynomials with respect either to  $a_s$  or  $b_s$ . The resultant has the merit to eliminate one of the variables and as result we have an univariate polynomial. Considering the resultant with respect to the coordinate  $a_s$  we find,

$$b_s \{ \varepsilon_2^4 + [\alpha_2 - \varepsilon_2^2(4\alpha_1 + \varepsilon_2^4)] b_s^4 - \alpha_1^2 b_s^8 \} = 0, \quad (\text{C.4})$$



where the dependence of the coefficients  $\alpha_1$  and  $\alpha_2$  on the parameters  $\varepsilon_1$  and  $\varepsilon_2$  are,

$$\alpha_1 = 4 + \varepsilon_2 [2\varepsilon_2 + \varepsilon_1^2\varepsilon_2 + \varepsilon_2^3 - \varepsilon_1(4 + \varepsilon_2^2)], \quad (\text{C.5})$$

$$\alpha_2 = -16 + 24\varepsilon_1\varepsilon_2 - 4(1 + 3\varepsilon_1^2)\varepsilon_2^2 + 2\varepsilon_1(2 + \varepsilon_1^2)\varepsilon_2^3 - (4 + \varepsilon_1^2)\varepsilon_2^4 + 2\varepsilon_1\varepsilon_2^5 + \varepsilon_2^6. \quad (\text{C.6})$$

The trivial solution  $b_s = 0$  to Eq.(C.4) implies also  $a_s = 0$  which has to be discarded since the point  $[0 : 0 : 1]$  does not belong to the curve  $C_1(a, b, 1)$ . This means that the allowed values for the coordinate  $b_s$  are the eight roots of the second factor of the polynomial (C.4). The corresponding coordinates for  $a_s$  can now be obtained by applying similar reasoning we used to intersect the surfaces in subsection 5.1. They can be expressed in terms of the variables  $b_s$  by the following expression,

$$a_s = \left[ \frac{b_s}{\varepsilon_2} \right]^3 \frac{\{\alpha_3 + [\varepsilon_2(\varepsilon_1 - \varepsilon_2) - 2]\alpha_1^2 b_s^4\}}{\alpha_1 + \varepsilon_2^2[2 - \varepsilon_2(\varepsilon_1 - \varepsilon_2)]}, \quad (\text{C.7})$$

where the coefficient  $\alpha_3$  is given by,

$$\begin{aligned} \alpha_3 = & -32 + 64\varepsilon_1\varepsilon_2 - 8(7 + 6\varepsilon_1^2)\varepsilon_2^2 + 4\varepsilon_1(21 + 4\varepsilon_1^2)\varepsilon_2^3 - 2(24 + 21\varepsilon_1^2 + \varepsilon_1^4)\varepsilon_2^4 \\ & + \varepsilon_1(48 + 7\varepsilon_1^2)\varepsilon_2^5 - 2(11 + 6\varepsilon_1^2)\varepsilon_2^6 + 11\varepsilon_1\varepsilon_2^7 - 5\varepsilon_2^8. \end{aligned} \quad (\text{C.8})$$

- The Curve  $Q_1(x, y, z)$

By solving the polynomial equations associated to the singular locus (78) of the target curve of the double cover map  $\phi$  we find nine singular points. Five of them are located on the affine plane and they are given by

$$P_A = [0 : 0 : 1] \text{ and } [x_s : y_s : 1], \quad (\text{C.9})$$

where the coordinates  $x_s$  and  $y_s$  satisfy the following decoupled equations,

$$\alpha_1^2 x_s^4 + \alpha_2 x_s^2 - \varepsilon_2^4 = 0, \quad (\text{C.10})$$

$$y_s = \frac{\varepsilon_2(2 - \varepsilon_1\varepsilon_2) - \varepsilon_2\alpha_1 x_s^2}{4 - 4\varepsilon_1\varepsilon_2 + \varepsilon_1^2\varepsilon_2^2 + \varepsilon_2^4}. \quad (\text{C.11})$$

The singularities at the infinity line  $z = 0$  sit on the same places of corresponding singular points associated to the domain degree eight curve, that is

$$P_\infty = [\pm \exp(i\frac{\pi}{3}) : 1 : 0], \quad [\pm \exp(i\frac{2\pi}{3}) : 1 : 0], \quad (\text{C.12})$$

except that now they behave as ordinary double points.

The only singularity that is not an ordinary double point turns out to be the one sited at the origin of the affine plane  $P_0 = [0 : 0 : 1]$ . The respective index of multiplicity is  $m_{P_0} = 4$  and it has an extra neighboring infinitesimal singularity. The desingularization diagram is thus given by,

$$\mathbf{Q}_1 \xrightarrow{\tilde{\pi}_2} \tilde{\mathbf{Q}}_1 \xrightarrow{\tilde{\pi}_1} \mathbf{Q}_1(a, b, c) \quad (\text{C.13})$$

where the curve  $\tilde{\mathbf{Q}}_1$  carries the infinitely near singularity associated to the point  $P_0 = [0 : 0 : 1]$  whose index of multiplicity is also four. By using this information we can easily obtain the corresponding genus of the normalization  $\mathbf{Q}_1$ , see Eq.(98).

## Appendix D: Reducible Curves

The purpose of this Appendix is to present the explicit expressions of the plane curves resulting from the intersection of the surfaces (80,81) when the parameters  $\varepsilon_1$  and  $\varepsilon_2$  are restricted to the submanifolds (100-102). We shall also see such degeneration gives origin to singular algebraic quartic curves with genus one.

- The linear submanifolds:

We first notice that it is enough to consider one of the submanifolds (100,101) because they are trivially related under the transformation  $\varepsilon_1 \rightarrow -\varepsilon_1$  and  $\varepsilon_2 \rightarrow -\varepsilon_2$ . In the case of the submanifold (100) we find, after eliminating the parameter  $\varepsilon_2$ , that the octic plane curve (87) becomes factorizable in terms of the following product,

$$C_1(a, b, c) = g(a, b, c)g(-a, b, ic) \quad (\text{D.1})$$

where the expression of the degree four plane curve  $g(a, b, c)$  is,

$$\begin{aligned} g(a, b, c) = & \left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2} + \varepsilon_1\right)a^4 \pm (-1 \mp i\sqrt{3} - \varepsilon_1 \pm i\sqrt{3}\varepsilon_1)a^3b + \left(\frac{3}{2} \pm i\frac{\sqrt{3}}{2} \mp i\sqrt{3}\varepsilon_1\right)a^2b^2 \\ & \pm (-2 + \varepsilon_1 \pm i\sqrt{3}\varepsilon_1)ab^3 + \left(\frac{1}{2} \mp i\frac{\sqrt{3}}{2} - \varepsilon_1\right)b^4 \pm (-1 \pm i\sqrt{3} + \varepsilon_1)a^2c^2 \\ & + \left(-2 + \frac{\varepsilon_1}{2} \pm i\frac{\sqrt{3}}{2}\varepsilon_1\right)abc^2 \pm \left(-1 \mp i\sqrt{3} - \frac{\varepsilon_1}{2} \pm i\frac{\sqrt{3}}{2}\varepsilon_1\right)b^2c^2 + \left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)c^4 \end{aligned} \quad (\text{D.2})$$

such that the symbol  $\pm$  refer to the two possible signs of the submanifold (100).

The plane curve (D.2) has only one singular point sited at the infinity line  $c = 0$  which behaves as a tacnode and as a consequence of that it has genus one. This means that there exists room for emerging new singularities for particular values of the parameter  $\varepsilon_1$  and an extra generation to a rational plane curve may be possible. Indeed, we find that this happens at the following values of the quadratic submanifolds (100,101),

$$\varepsilon_1 = \varepsilon_2 = \pm 2, \quad \text{and} \quad \varepsilon_1 = -\varepsilon_2 = \pm \frac{2i}{\sqrt{3}}, \quad (\text{D.3})$$

where now the quartic plane curves factorizes once again in the product of two conics. As a result, the original octic plane curve (87) degenerates to genus zero curve and the corresponding weights of the vertex model becomes trigonometric at the specific points (D.3).

- The quartic submanifold:

The essential ingredient in the analysis of the third submanifold (102) is to note that we are in fact dealing with a curve that can be rationally parameterized. In fact, one easily finds that submanifold (102) has the maximal number of three ordinary singular points when viewed in the projective space. Now given such three double points on a quartic curve we can fix any other point on it and pass conics through these four points. By using this family of conics it is well known that one can establish a rational parameterization and the final result is,

$$\varepsilon_1 = -\frac{4(-3 + 14\kappa)(147t^2 - 98t + 294\kappa t + 12 - 46\kappa)(-49t^2 + 98\kappa t + 2 - 24\kappa)}{(133t^2 - 56t + 84\kappa t + 4 - 6\kappa)(343t^2 - 196t + 980\kappa t + 24 - 190\kappa)} \quad (\text{D.4})$$

$$\varepsilon_2 = -\frac{2(-4 + 63\kappa)(147t^2 - 98t + 294\kappa t + 12 - 46\kappa)(637t^2 - 294t + 490\kappa t + 36 - 138\kappa)}{13(133t^2 - 56t + 84\kappa t + 4 - 6\kappa)(343t^2 - 196t + 980\kappa t + 24 - 190\kappa)} \quad (\text{D.5})$$

where  $t$  is the parameterization variable and  $\kappa$  is a constant factor  $\kappa = \frac{6 \pm i\sqrt{13}}{49}$ .

With the help of the parameterization (D.4) we then are able to investigate the explicit factorization of the octic plane curve (87) in terms of the product of two quartic curves, namely

$$C_1(a, b, c) = h(a, b, c)h(-a, b, ic) \quad (\text{D.6})$$

where the expression for  $h(a, b, c)$  is,

$$\begin{aligned}
h(a, b, c) &= \frac{\kappa_1}{109531219}(a^4 + b^4 + a^2b^2) + \frac{(29 \pm 36i\sqrt{13})\kappa_2}{14567652127}c^4 \mp \frac{2(\pm 2i\sqrt{13} + 9)\kappa_3^2}{6900466797}a^2c^2 \\
&\pm \frac{2(\pm 2i\sqrt{13} + 9)\kappa_4^2}{1684480617001}b^2c^2 \mp \frac{2(\pm 9i\sqrt{13} - 26)\kappa_3\kappa_4}{388726296231}abc^2
\end{aligned} \tag{D.7}$$

such that the coefficients  $\kappa_1, \dots, \kappa_4$  are given by,

$$\begin{aligned}
\kappa_1 &= (45619t^2 - 16856t \pm 392i\sqrt{13}t + 1836 \mp 86i\sqrt{13})(160 - 1372t + 2401t^2 \mp 6i\sqrt{13}) \\
\kappa_2 &= (16807t^2 \pm 980i\sqrt{13}t - 3724t \mp 190i\sqrt{13} + 36)(160 - 2240t \pm 84i\sqrt{13}t + 6517t^2 \mp 6i\sqrt{13}) \\
\kappa_3 &= (147t \pm 3i\sqrt{13} - 31 \mp 12i + 2\sqrt{13})(147t - 31 \pm 12i - 2\sqrt{13} \pm 3i\sqrt{13}) \\
\kappa_4 &= (637t \pm 5i\sqrt{13} - 117 \mp 26i - 12\sqrt{13})(637t \pm 5i\sqrt{13} - 117 \pm 26i + 12\sqrt{13})
\end{aligned} \tag{D.8}$$

For generic values of the free variable  $t$  one finds that the quartic plane curve (D.7) has two ordinary singular points and thus has again genus one. We remark that in this submanifold we have not been able to find a further generation to rational curves. This however can not be ruled out since in this case the analysis is more subtle.

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